Problem 2 Black Hole in Milky Way

The astronomers have been surprised by some observational facts:

1) The intensity of the light coming from the center of the Milky Way is smaller than the intensity of the light coming from the stars around our galaxy;
2) The stars located close to the center of the Milky Way change its apparent position faster than those located far away from the center of the galaxy.
   To justify those observational facts, the scientists admit the presence of a black-hole in the center of Milky Way.

In the center of Milky Way, is located the hypothetical black-hole (Sagittarius A*). A star S* is orbiting the black-hole A*

In the table 1 the following data is presented : the date and the angular position coordinates $(\alpha;\beta)$ of the star S* at different moments of the observation. The coordinates represent the angular distances of the projection or the star S* in the coordinates system (U,W) , centered on the Sagittarius A* see figure 1.

An angular distance $\varphi = 1\,\text{arcsec}$ corresponds a linear distance in the sky -plane $d = 41\,\text{light day}$, so to a scale $S_0 = \dfrac{d}{\varphi} = 41\,\dfrac{\text{light day}}{\text{arcsec}}$.

	Date (year)	$\alpha(\text{arcsec})$	$\beta(\text{arcsec})$
1	1995.222	0.117	- 0.166
2	1997.526	0.097	- 0.189
3	1998.326	0.087	- 0.192
4	1999.041	0.077	- 0.193
5	2000.414	0.052	- 0.183
6	2001.169	0.036	- 0.167
7	2002.831	- 0.000	- 0.120
8	2003.584	- 0.016	- 0.083
9	2004.165	- 0.026	- 0.041
10	2004.585	- 0.017	0.008
11	2004.655	- 0.004	0.014
12	2004.734	0.008	0.017
13	2004.839	0.021	0.012
14	2004.936	0.037	0.009
15	2005.503	0.072	- 0.024
16	2006.041	0.088	- 0.050
17	2007.060	0.108	- 0.091

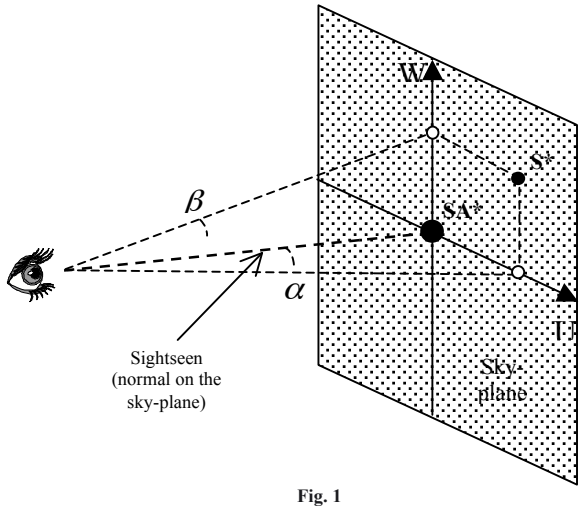


Fig. 1

By using the information provided your tasks are:

a) Plot the projection of the trajectory of the star S* in the plane P (see the figure 2). This plane is close to the observer. In this plane to $\varphi = 1\,\text{arcsec}$ correspond a linear distance d_0= 1200 mm so the scale is $S = \frac{d_0}{\varphi} = 1200\,\frac{mm}{\text{arcsec}}$. You have to use the milimetric paper, carbon copy sheet of paper and the transparent sheets for an accurate plot.

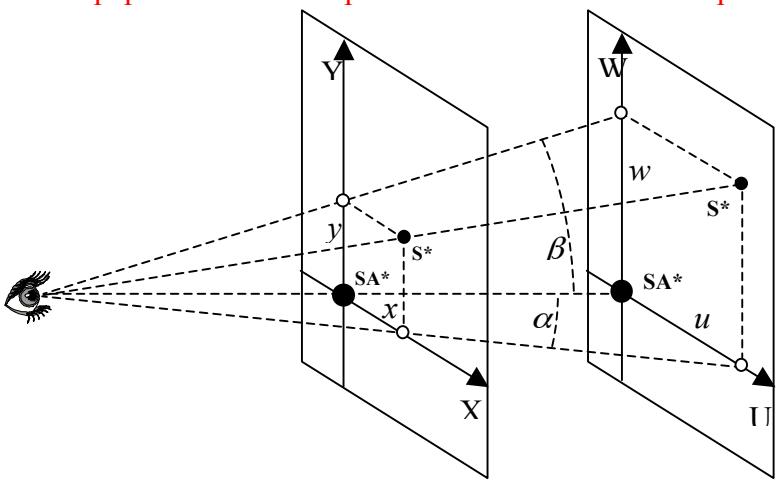


Fig. 2

b) By using the plot prove that the sight seen direction is normal to the plane of the orbit

c) Using your plot find out following elements of the real orbit of star S* around the black hole Sagittarius A*:

I. a – large semi-axis (in light days units) ; b – small semi-axis in (in light days units); e – eccentricity;

II. $r_{\min}$ – the minimum distance between S* and SA*(in light days units); $r_{\max}$ – the maximum distance between S* and SA* (in light days units);

III. The distance from the observer to the S*;

IV. the orbiting period of star S* around Sagittarius A* (will be accepted as a correct result the arithmetic mean of the maximum number of findings);

V. the total mass of the system "Sagitarius A* - Star S*".

Presenting the intermediate and final data in tables is recommended for an accurate evaluation.

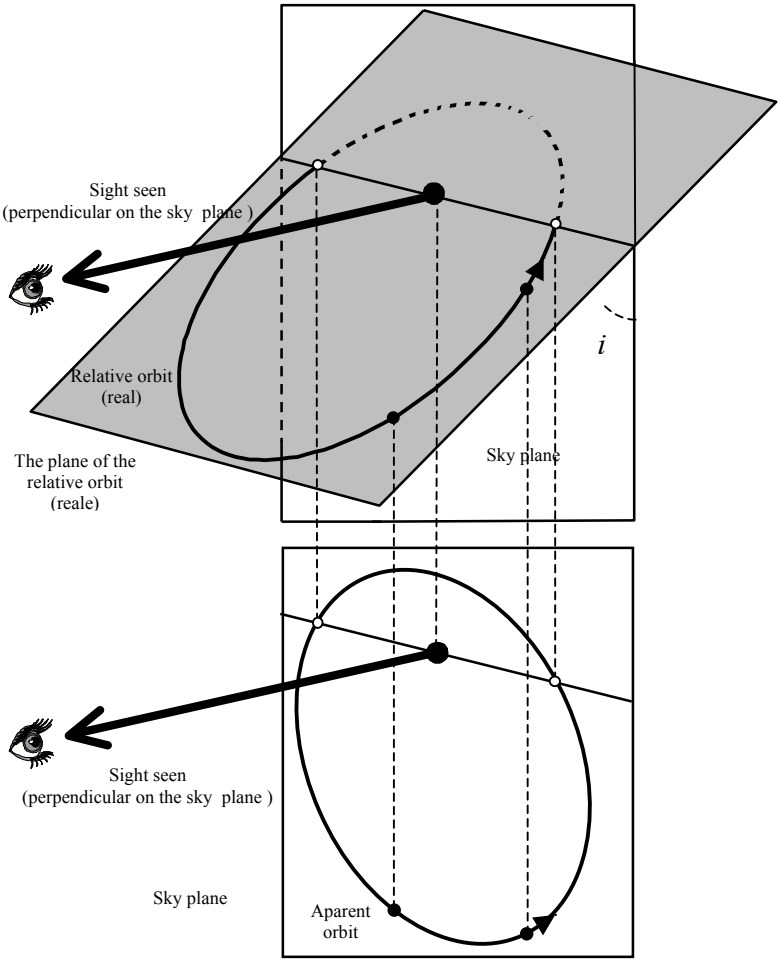
$G = 6{,}67 \cdot 10^{-11}\ \mathrm{Nm^2kg^{-2}}$

Problem 2. Marking scheme Black Hole in Milky Way

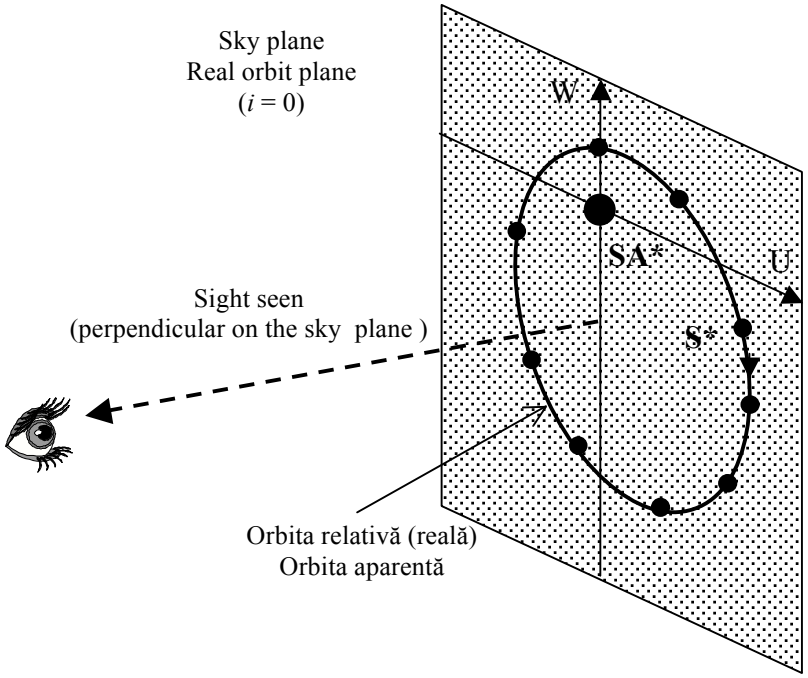
- **a)**	**15 points**
- **b)**	**10 points**
- **c)**	**25 points**
○ I	5 points
○ II	5 points
○ III	5 points
○ IV	5 points
○ V	5 points

Detailed solution

Usually the plane of apparent orbit (sky plane) is different from the plane of the relative orbit due to the angle between the two planes ($i = 0$).



In the case of the problem the sky plane and the plane of the relative orbit are the same ($i = 0$), See the figure.



The linear position coordinates (u, w), of the projection of the star S* in the sky plane with the coordinate axes (U, W), having the origin in Sagittarius A*, expressed in light-days are given in the table 1. (for an angular distance $\varphi = 1\,\text{arcsec}$ correspond in sky plane the linear distance $d = 41\,\text{light day}$).

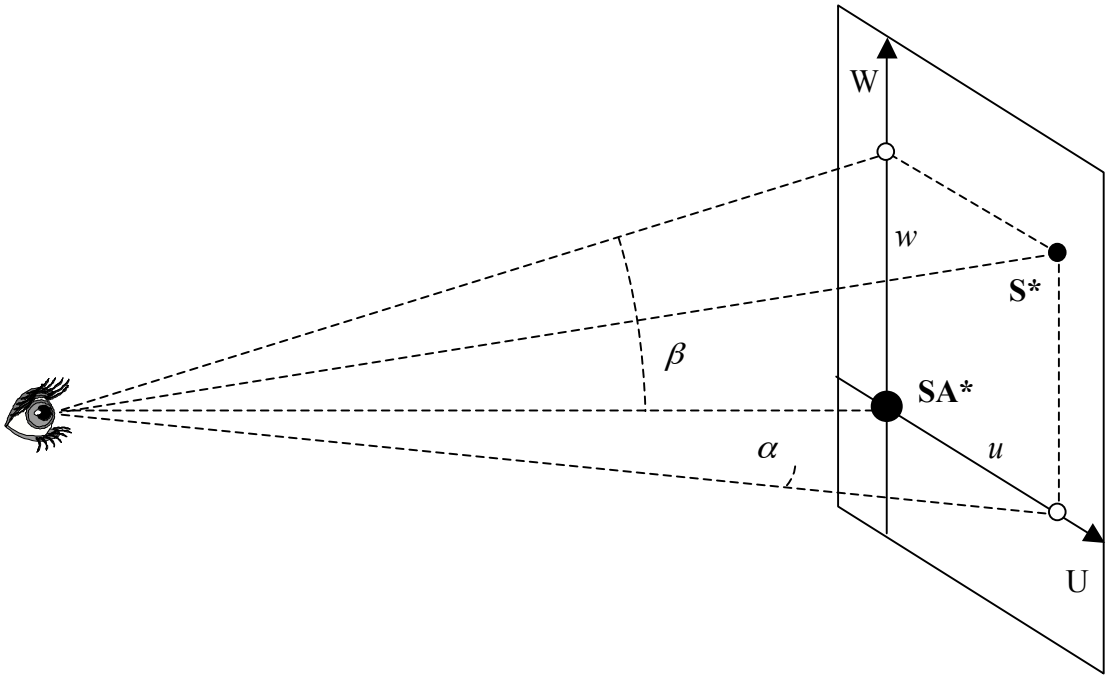


Table 1 The coordinates in light-days

	Date (year)	α(arcsec)	u(ld)	β(arcsec)	w(ld)
1	1995.222	0.117	4.797	- 0.166	-6.806
2	1997.526	0.097	3.977	- 0.189	-7.749
3	1998.326	0.087	3.567	- 0.192	-7.872
4	1999.041	0.077	3.157	- 0.193	-7.913
5	2000.414	0.052	2.132	- 0.183	-7.503
6	2001.169	0.036	1.476	- 0.167	-6.847
7	2002.831	- 0.000	0.000	- 0.120	-4,920
8	2003.584	- 0.016	-0.656	- 0.083	-3.403
9	2004.165	- 0.026	-1.066	- 0.041	-1.681
10	2004.585	- 0.017	-0.697	0.008	0.328
11	2004.655	- 0.004	-0.164	0.014	0.574
12	2004.734	0.008	0.328	0.017	0.697
13	2004.839	0.021	0.861	0.012	0.492
14	2004.936	0.037	1.517	0.009	0.369
15	2005.503	0.072	2.952	- 0.024	-0.984
16	2006.041	0.088	3.608	- 0.050	-2.050
17	2007.060	0.108	4.428	- 0.091	-3.731

To represent the image of the relative orbit of star S* in a plane parallel with the sky plane we use a scale of $S = 1200 \frac{\text{mm}}{\text{arcsec}}$, and for conversion we use $x = \alpha S$ and $y = \beta S$, see figure 3. These coordinates of the image of the star S* relative to the origin of the axes – the image of Sagittarius A* are given in the below table 3

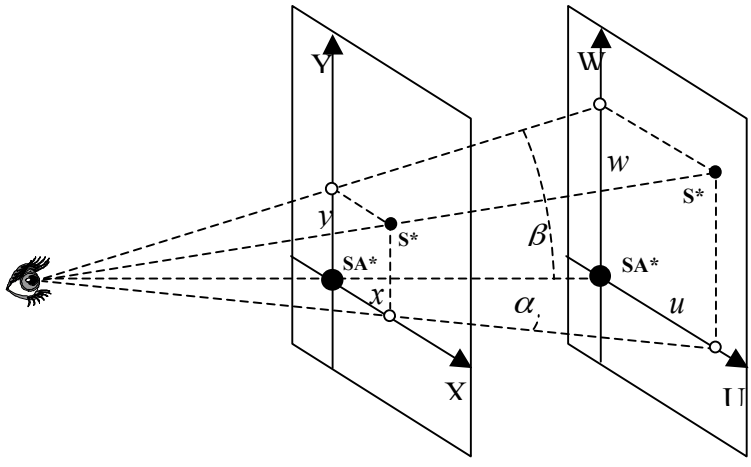


Table 2 The coordinates in scaled plane

	Date (year)	$\alpha(\text{arcsec})$	$x(\text{mm})$	$\beta(\text{arcsec})$	$y(\text{mm})$
1	1995.222	0.117	140.4	- 0.166	-199.2
2	1997.526	0.097	116.4	- 0.189	-226.8
3	1998.326	0.087	104.4	- 0.192	-230.4
4	1999.041	0.077	92.4	- 0.193	-231.6
5	2000.414	0.052	62.4	- 0.183	-219.6
6	2001.169	0.036	43.2	- 0.167	-200.4
7	2002.831	- 0.000	0.0	- 0.120	-144.0
8	2003.584	- 0.016	-20.0	- 0.083	-100.0
9	2004.165	- 0.026	-32.0	- 0.041	-50.0
10	2004.585	- 0.017	-20.4	0.008	9.6
11	2004.655	- 0.004	-4.8	0.014	16.8
12	2004.734	0.008	9.6	0.017	20.4
13	2004.839	0.021	25.2	0.012	14.4
14	2004.936	0.037	44.4	0.009	10.8
15	2005.503	0.072	86.4	- 0.024	-28.8
16	2006.041	0.088	106.0	- 0.050	-60.0
17	2007.060	0.108	130.0	- 0.091	-110.0

The following pictures represents the plot of the values in the table 2, and the measurements which have to be done in order to find out the required data.

Numeric processing

a) From the measurements done on paper system (X, Y):

$$a_{maesured} \approx 135\,\text{mm}; b_{mesured} \approx 66\,\text{mm}; d_{\min} = 17\,\text{mm},$$

Corresponding to the numeric scale:

$$a_{mesured} = \varphi_a \cdot S; b_{mesured} = \varphi_{b_{,,}} \cdot S; d_{\min\, mes} = \varphi_{\min} \cdot S,$$

$$\varphi_a = 0.1125\,\text{arcsec};$$

$$\varphi_b = 0.055\,\text{arcsec};$$

$$\varphi_{d\min} = 0{,}01416\,\text{arcsec};$$

$$1\,\text{arcsec} \ldots\ldots\ldots\ldots\ldots\ldots\ldots 41\,\text{light day};$$

$$\varphi_a = 0.1125\,\text{arcsec} \ldots\ldots\ldots\ldots a;$$

$$\varphi_b = 0.055\,\text{arcsec} \ldots\ldots\ldots\ldots b;$$

$$\varphi_{d\min} = 0{,}014666\,\text{arcsec} \ldots\ldots\ldots\ldots r_{\min};$$

$$a = \frac{0.1125\,\text{arcsec} \cdot 41\,\text{light days}}{1\,\text{arcsec}};\ a = 4.6125\,\text{light days},$$

$$b = \frac{0.055\,\text{arcsec} \cdot 41\,\text{light days}}{1\,\text{arcsec}};\ b = 2.255\ \text{light days},$$

$$r_{\min} = \frac{0{,}0146\,\text{arcsec} \cdot 41\,\text{light days}}{1\,\text{arcsec}} = 0{,}5986\ \text{light days};$$

$$c = \sqrt{a^2 - b^2} \approx 4{,}0236\,\text{light days};$$

$$r_{\min} = a - c = 0{,}5889\,\text{light days};\ r_{\max} = a + c = 8{,}6361\,\text{light days};$$

$$e = \sqrt{1 - \frac{b^2}{a^2}} \approx 0.87,$$

b) $S_{\text{elipsa}} = 28560\,\text{mm}^2$; See Anex

$$T \ldots\ldots\ldots\ldots\ldots\ldots\ldots S_{\text{elipsa}};$$

$$\Delta t \ldots\ldots\ldots\ldots\ldots\ldots \Delta S;$$

$$T = \frac{S_{\text{elipsa}}}{\Delta S} \cdot \Delta t;$$

Displacement $1 \rightarrow 2$

$$\Delta S_{12} = \frac{252\,\text{mm x } 34\,\text{mm}}{2} + 86\,\text{mm}^2 \approx 4370\,\text{mm}^2;$$

$$t_1 = 1995.222\,\text{years}; t_2 = 1997.526\,\text{years};$$

$$\Delta t_{12} = t_2 - t_1 = 2.304\,\text{years};$$

$$T \ldots\ldots\ldots\ldots\ldots 28560\,\text{mm}^2;$$

$$2.304\,\text{years} \ldots\ldots\ldots\ldots\ldots 4370\,\text{mm}^2;$$

$$T = 15.057\,\text{years};$$

Displacement $2 \rightarrow 3$

$$\Delta S_{23} = \frac{252\,\text{mm x } 12\,\text{mm}}{2} + 20\,\text{mm}^2 \approx 1532\,\text{mm}^2;$$

$$t_2 = 1997.526\,\text{years}; t_3 = 1998.326\,\text{years};$$

$$\Delta t_{23} = t_3 - t_2 = 0.800\,\text{years};$$

$$T \ldots\ldots\ldots\ldots\ldots 28560\,\text{mm}^2;$$

$$0.800\,\text{years} \ldots\ldots\ldots\ldots\ldots 1532\,\text{mm}^2;$$

$$T = 14.913\,\text{years};$$

Displacement $3 \rightarrow 4$

$$\Delta S_{34} = \frac{250\,\text{mm x } 11\,\text{mm}}{2} + 26\,\text{mm}^2 \approx 1400\,\text{mm}^2;$$

$$t_3 = 1998.326\,\text{years}; t_4 = 1999.041\,\text{years};$$

$$\Delta t_{34} = t_4 - t_3 = 0.715\,\text{years};$$

$$T \ldots\ldots\ldots\ldots\ldots 28560\,\text{mm}^2;$$

$$0.715\,\text{years} \ldots\ldots\ldots\ldots\ldots 1400\,\text{mm}^2;$$

$$T = 14.586 \text{ years};$$

Displacement $4 \rightarrow 5$

$$\Delta S_{45} = \frac{244 \text{ mm x } 22 \text{ mm}}{2} + 30 \text{ mm}^2 = 2714 \text{ mm}^2;$$

$$t_4 = 1999.041 \text{ years}; t_5 = 2000.414 \text{ years};$$

$$\Delta t_{45} = t_5 - t_4 = 1.373 \text{ years};$$

$$T \text{ ..... } 28560 \text{ mm}^2;$$

$$1.373 \text{ years ..... } 2714 \text{ mm}^2;$$

$$T = 14.448 \text{ years};$$

Displacement $5 \rightarrow 6$

$$\Delta S_{56} = \frac{220 \text{ mm x } 13 \text{ mm}}{2} + 5 \text{ mm}^2 = 1435 \text{ mm}^2;$$

$$t_5 = 2000.414 \text{ years}; t_6 = 2001.169 \text{ years};$$

$$\Delta t_{56} = 0.755 \text{ years};$$

$$T \text{ ..... } 28560 \text{ mm}^2;$$

$$0.755 \text{ years ..... } 1435 \text{ mm}^2;$$

$$T = 15.026 \text{ years};$$

Displacement $6 \rightarrow 7$

$$\Delta S_{67} \approx \frac{203 \text{ mm x } 30 \text{ mm}}{2} + 120 \text{ mm}^2 \approx 3165 \text{ mm}^2;$$

$$t_6 = 2001.169 \text{ years}; \ t_7 = 2002.831 \text{ years};$$

$$\Delta t_{67} = 1.662 \text{ years};$$

$$T \text{ ..... } 28560 \text{ mm}^2;$$

$$1.662 \text{ years ..... } 3165 \text{ mm}^2;$$

$$T = 14.997 \text{ years};$$

Displacement $7 \rightarrow 8$

$$\Delta S_{78} \approx \frac{143 \text{ mm x } 20 \text{ mm}}{2} + 5 \text{ mm}^2 \approx 1435 \text{ mm}^2;$$

$$t_7 = 2002.831 \text{ years}; t_8 = 2003.584 \text{ years};$$

$$\Delta t_{78} = 0.753 \text{ years};$$

$$T \text{ ..... } 28560 \text{ mm}^2;$$

$$0.753 \text{ years ..... } 1435 \text{ mm}^2;$$

$$T = 14.986 \text{ years};$$

Displacement $8 \rightarrow 9$

$$\Delta S_{89} \approx \frac{100 \text{ mm x } 22 \text{ mm}}{2} + 7 \text{ mm}^2 \approx 1107 \text{ mm}^2;$$

$$t_8 = 2003.584 \text{ years}; t_9 = 2004.165 \text{ years};$$

$$\Delta t_{89} = 0.581 \text{years};$$

$$T \ \ldots\ldots\ldots\ldots\ldots\ 28560\ \text{mm}^2;$$

$$0.581\ \text{years} \ldots\ldots\ldots\ldots\ldots\ 1107\ \text{mm}^2;$$

$$T = 14.989\ \text{years};$$

Displacement $9 \rightarrow 10$

$$\Delta S_{9;10} \approx \frac{60\ \text{mm x}\ 20\ \text{mm}}{2} + 200\ \text{mm}^2 \approx 800\ \text{mm}^2;$$

$$t_9 = 2004.165\ \text{years}; t_{10} = 2004.585\ \text{years};$$

$$\Delta t_{9,10} \approx 0.420\ \text{years};$$

$$T \ \ldots\ldots\ldots\ldots\ldots\ 28560\ \text{mm}^2;$$

$$0.420\ \text{years} \ldots\ldots\ldots\ldots\ldots\ 800\ \text{mm}^2;$$

$$T = 14.994\ \text{years};$$

Displacement $10 \rightarrow 11$

$$\Delta S_{10;11} \approx \frac{20\ \text{mm x}\ 13\ \text{mm}}{2} + 5\ \text{mm}^2 \approx 135\ \text{mm}^2;$$

$$t_{10} = 2004.585\ \text{years}; t_{11} = 2004.655\ \text{years};$$

$$\Delta t_{10;11} \approx 0.070\ \text{years};$$

$$T \ \ldots\ldots\ldots\ldots\ldots\ 28560\ \text{mm}^2;$$

$$0{,}070\ \text{years} \ldots\ldots\ldots\ldots\ldots\ 135\ \text{mm}^2;$$

$$T = 14.808\ \text{years};$$

Displacement $11 \rightarrow 12$

$$\Delta S_{11;12} = \frac{22\ \text{mm x}\ 13\ \text{mm}}{2} + 4\ \text{mm}^2 = 147\ \text{mm}^2;$$

$$t_{11} = 2004.655\ \text{years}; t_{12} = 2004.734\ \text{years};$$

$$\Delta t_{11;12} \approx 0.077\ \text{years};$$

$$T \ \ldots\ldots\ldots\ldots\ldots\ 28560\ \text{mm}^2;$$

$$0{,}077\ \text{years}; \ldots\ldots\ldots\ldots\ldots\ 147\ \text{mm}^2;$$

$$T = 14.960\ \text{years};$$

Displacement $12 \rightarrow 13$

$$\Delta S_{12;13} = \frac{30\ \text{mm x}\ 13\ \text{mm}}{2} + 4\ \text{mm}^2 = 200\ \text{mm}^2;$$

$$t_{12} = 2004.734\ \text{years}; t_{13} = 2004.839\ \text{years};$$

$$\Delta t_{12;13} \approx 0.105\ \text{years};$$

$$T \ \ldots\ldots\ldots\ldots\ldots\ 28560\ \text{mm}^2;$$

$$0{,}105\ \text{years} \ldots\ldots\ldots\ldots\ldots\ 200\ \text{mm}^2;$$

$$T = 14.994\ \text{years};$$

Displacement $13 \rightarrow 14$

$$\Delta S_{13;14} = \frac{41\,\text{mm x } 9\,\text{mm}}{2} \approx 185\,\text{mm}^2;$$

$$t_{13} = 2004.839\ \text{years};\ t_{14} = 2004.936\ \text{years};$$

$$\Delta t_{13;14} \approx 0.097\ \text{years};$$

$$T \ \ldots\ldots\ldots\ldots\ldots\ 28560\,\text{mm}^2;$$

$$0.097\ \text{years} \ldots\ldots\ldots\ldots\ldots\ 185\,\text{mm}^2;$$

$$T = 14.974\ \text{years};$$

Displacement $14 \rightarrow 15$

$$\Delta S_{14;15} = \frac{90\,\text{mm x } 22\,\text{mm}}{2} + 90\,\text{mm}^2 = 1080\,\text{mm}^2;$$

$$t_{14} = 2004.936\ \text{years};\ t_{15} = 2005.503\ \text{years};$$

$$\Delta t_{14;15} \approx 0.567\ \text{years};$$

$$T \ \ldots\ldots\ldots\ldots\ldots\ 28560\,\text{mm}^2;$$

$$0.567\ \text{years} \ldots\ldots\ldots\ldots\ldots\ 1080\,\text{mm}^2;$$

$$T = 14.994\ \text{years};$$

Displacement $15 \rightarrow 16$

$$\Delta S_{15;16} = \frac{1}{2} \cdot 120\,\text{mm x } 17\,\text{mm} + 5\,\text{mm}^2 = 1025\,\text{mm}^2;$$

$$t_{15} = 2005.503\ \text{years};\ t_{16} = 2006.041\ \text{years};$$

$$\Delta t_{15;16} \approx 0.538\ \text{years};$$

$$T \ \ldots\ldots\ldots\ldots\ldots\ 28560\,\text{mm}^2;$$

$$0{,}538\ \text{years} \ldots\ldots\ldots\ldots\ldots\ 1025\,\text{mm}^2;$$

$$T = 14.990\ \text{years};$$

Displacement $16 \rightarrow 17$

$$\Delta S_{16;17} = \frac{168\,\text{mm x } 23\,\text{mm}}{2} + 10\,\text{mm}^2 = 1942\,\text{mm}^2;$$

$$t_{16} = 2006.041\ \text{years};\ t_{17} = 2007.060\ \text{years};$$

$$\Delta t_{16;17} = 1.019\ \text{years};$$

$$T \ \ldots\ldots\ldots\ldots\ldots\ 28560\,\text{mm}^2;$$

$$1.019\ \text{years} \ldots\ldots\ldots\ldots\ldots\ 1942\,\text{mm}^2;$$

$$T = 14.985\ \text{years};$$

Displacement $17 \rightarrow 1$

$$\Delta S_{17;1} = \frac{240\,\text{mm x } 43\,\text{m}}{2} + 300\,\text{mm}^2 = 5460\,\text{mm}^2;$$

$$t_{17} = 2007.060\ \text{years};$$

$$\Delta t_{17;1} = 2{,}867\ \text{years};$$

$$T \ \ldots\ldots\ldots\ldots\ldots\ 28560\,\text{mm}^2;$$

$$2{,}867 \text{ years} \ldots\ldots\ldots\ldots\ldots\ldots\ldots 5460\,\text{mm}^2;$$

$$T = 14.996 \text{ years};$$

$$\overline{T} = 14.923 \text{ years};$$

$$T \approx 15 \text{ years}.$$

c) By using the III rd Kepler's law:

$$T^2 = \frac{4\pi^2}{K(M_{SA^*} + M_{S^*})} \cdot a^3;$$

$$M_{SA^*} + M_{S^*} = \frac{4\pi^2 a^3}{KT^2};$$

$$a = 4.6125 \text{ light days};\ T \approx 15 \text{ years};\ K = 6{,}67 \cdot 10^{-11}\ \text{Nm}^2\text{kg}^{-2};$$

$$1\,\text{ld} \approx 2{,}6 \cdot 10^{13}\ \text{m};\ 1 \text{ years} \approx 3{,}1 \cdot 10^7\ \text{s};\ 4\pi^2 \approx 40;$$

$$(2{,}6)^3 \approx 17{,}6;\ (4{,}6125)^3 \approx 10^2;$$

$$M_{SA^*} + M_{S^*} = \frac{40 \cdot 10^2 \cdot 17{,}6 \cdot 10^{39}}{6{,}67 \cdot 10^{-11} \cdot 225 \cdot 10^{15}}\ \text{kg};$$

$$M_{SA^*} + M_{S^*} \approx 4{,}6 \cdot 10^{36}\ \text{kg};$$

$$M_{Sun} \approx 1{,}8 \cdot 10^{30}\ \text{kg};$$

$$M_{SA^*} + M_{S^*} \approx 2{,}5 \cdot 10^6 \cdot M_{Sun};$$

$$M_{S^*} << M_{SA^*};$$

$$M_{SA^*} \approx 2{,}5 \cdot 10^6 \cdot M_{Sun},$$