(D2) **Distance to the Moon**

Geocentric ephemerides of the Moon for September 2015 are given in the form of a table. Each reading was taken at 00:00 UT.

Date	R.A. (α)			Dec. (δ)			Angular Size (θ)	Phase (ϕ)	Elongation of Moon
	h	m	s	$^\circ$	$'$	$''$	$''$		
Sep 01	0	36	46.02	3	6	16.8	1991.2	0.927	148.6° W
Sep 02	1	33	51.34	7	32	26.1	1974.0	0.852	134.7° W
Sep 03	2	30	45.03	11	25	31.1	1950.7	0.759	121.1° W
Sep 04	3	27	28.48	14	32	4.3	1923.9	0.655	107.9° W
Sep 05	4	23	52.28	16	43	18.2	1896.3	0.546	95.2° W
Sep 06	5	19	37.25	17	55	4.4	1869.8	0.438	82.8° W
Sep 07	6	14	19.23	18	7	26.6	1845.5	0.336	70.7° W
Sep 08	7	7	35.58	17	23	55.6	1824.3	0.243	59.0° W
Sep 09	7	59	11.04	15	50	33.0	1806.5	0.163	47.5° W
Sep 10	8	49	0.93	13	34	55.6	1792.0	0.097	36.2° W
Sep 11	9	37	11.42	10	45	27.7	1780.6	0.047	25.1° W
Sep 12	10	23	57.77	7	30	47.7	1772.2	0.015	14.1° W
Sep 13	11	9	41.86	3	59	28.8	1766.5	0.001	3.3° W
Sep 14	11	54	49.80	0	19	50.2	1763.7	0.005	7.8° E
Sep 15	12	39	50.01	-3	20	3.7	1763.8	0.026	18.6° E
Sep 16	13	25	11.64	-6	52	18.8	1767.0	0.065	29.5° E
Sep 17	14	11	23.13	-10	9	4.4	1773.8	0.120	40.4° E
Sep 18	14	58	50.47	-13	2	24.7	1784.6	0.189	51.4° E
Sep 19	15	47	54.94	-15	24	14.6	1799.6	0.270	62.5° E
Sep 20	16	38	50.31	-17	6	22.8	1819.1	0.363	73.9° E
Sep 21	17	31	40.04	-18	0	52.3	1843.0	0.463	85.6° E
Sep 22	18	26	15.63	-18	0	41.7	1870.6	0.567	97.6° E
Sep 23	19	22	17.51	-17	0	50.6	1900.9	0.672	110.0° E
Sep 24	20	19	19.45	-14	59	38.0	1931.9	0.772	122.8° E
Sep 25	21	16	55.43	-11	59	59.6	1961.1	0.861	136.2° E
Sep 26	22	14	46.33	-8	10	18.3	1985.5	0.933	150.0° E
Sep 27	23	12	43.63	-3	44	28.7	2002.0	0.981	164.0° E
Sep 28	0	10	48.32	0	58	58.2	2008.3	1.000	178.3° E
Sep 29	1	9	5.89	5	38	54.3	2003.6	0.988	167.4° W
Sep 30	2	7	39.02	9	54	16.1	1988.4	0.947	153.2° W

The composite graphic[1] below shows multiple snapshots of the Moon taken at different times during the total lunar eclipse, which occurred in this month. For each shot, the centre of frame was coinciding with the central north-south line of umbra.

For this problem, assume that the observer is at the centre of the Earth and angular size refers to angular diameter of the relevant object / shadow.



(D2.1) In September 2015, apogee of the lunar orbit is closest to **3**
New Moon / First Quarter / Full Moon / Third Quarter.
Tick the correct answer in the Summary Answersheet. No justification for your answer is necessary.

Solution:
From the table we see that the angular size of Moon is smallest close to the New Moon day. Thus, the answer is New Moon. **3.0**
Justification is NOT necessary for full credit.

(D2.2) In September 2015, the ascending node of lunar orbit with respect to the ecliptic is closest to **4**
New Moon / First Quarter / Full Moon / Third Quarter.
Tick the correct answer in the Summary Answersheet. No justification for your answer is necessary.

Solution:
As there is an eclipse happening in this month, the lunar nodes are close to Full Moon day and New Moon day. Next we notice that lowest declination of Moon is just 18°. This means that after the New Moon day, the orbit of Moon is above the ecliptic. In other words, the ascending node is near the New Moon. **4.0**
Justification is NOT necessary for full credit.

(D2.3) Estiamte the eccentricity, e, of the lunar orbit from the given data. **4**

Solution:
The largest angular size of the Moon in the ephemerides is $2008.3''$ and the smallest angular size is $1763.7''$. The distance is inversely proportional to the angular size. Hence ratio of distance at perigee to the distance at apogee is:

$$\text{Ratio} = \frac{r_{\text{perigee}}}{r_{\text{apogee}}} = \frac{\cancel{a}}{1+e} \times \frac{1-e}{\cancel{a}}$$ **2.0**

[1]Credit: NASA's Scientific Visualization Studio

$$\therefore \frac{1-e}{1+e} = \frac{1763.7}{2008.3} = 0.87821$$

$$\therefore e = \frac{1-0.87821}{1+0.87821} = 0.064846$$

$$\boxed{e \simeq 0.065}$$

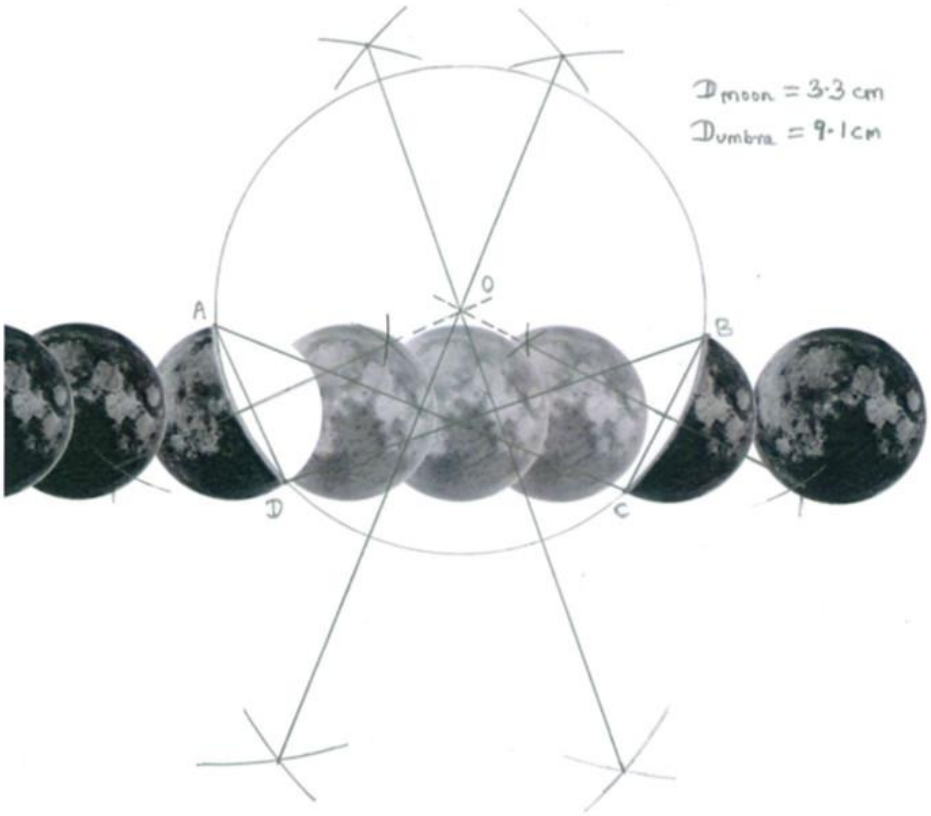
2.0

Rounding off is done to account for the fact that our data are not continuous, hence exact angular sizes at perigee and apogee are not known. A non-rounded answer will also receive full credit.

(D2.4) Estimate the angular size of the umbra, θ_{umbra}, in terms of the angular size of the Moon, θ_{Moon}. Show your working on the image given on the backside of the Summary Answersheet. **8**

Solution:

The following construction is shown. **5.0**



Only two chords are necessary to determine the centre.

The credit is divided in two parts for the drawing:

- **Realisation that centre of umbra circle needs to be determined to find θ_{umbra}: 1.5**
- **Accurate determination of centre of umbra circle by geometric construction: 2.5**
  Determination of centre of hand-drawn circle: maximum 1.5

- **Measuring diameters of umbra and Moon: 0.5 each**

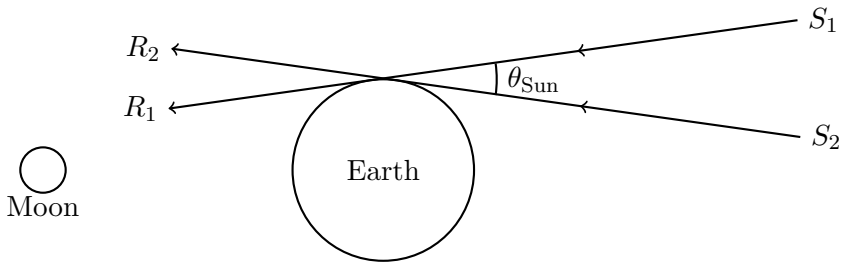
By estimating approximate centre of the shadow in the image, we find out,

$$\frac{\theta_{\text{umbra}}}{\theta_{\text{Moon}}} = \frac{d_{\text{umbra}}}{d_{\text{Moon}}} = \frac{9.1}{3.3} = 2.76 \quad \textbf{2.0}$$

$$\therefore \boxed{\theta_{\text{umbra}} \simeq 2.76\theta_{\text{Moon}}} \quad \textbf{1.0}$$

Acceptable range: ± 0.10**.**

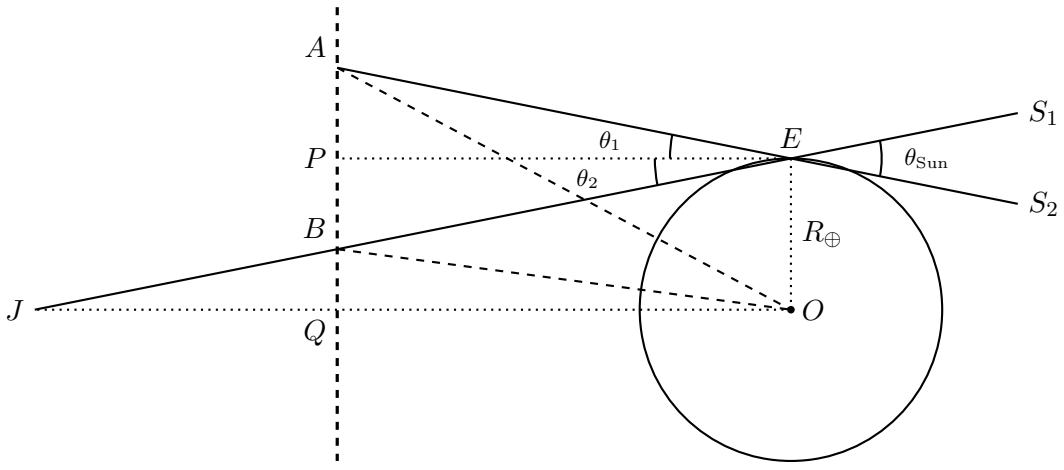
(D2.5) The angle subtended by the Sun at Earth on the day of the lunar eclipse is known to be $\theta_{\text{Sun}} = 1915.0''$. In the figure below, S_1R_1 and S_2R_2 are rays coming from diametrically opposite ends of the solar disk. The figure is not to scale. **9**



Calculate the angular size of the penumbra, θ_{penumbra}, in terms of θ_{Moon}. Assume the observer to be at the centre of the Earth.

Solution:

The following diagram needs to be drawn.



Moon's path

Angular size of umbra is $\theta_{\text{umbra}} = 2\angle BOQ$
Angular size of penumbra is $\theta_{\text{penumbra}} = 2\angle AOQ$
We have

$$QA = QP + PA$$
$$= OE + PE\tan\theta_1$$

$$\approx R_\oplus + d_{\text{Moon}}\theta_1 \qquad \text{since } PA \ll PE$$

$$\approx R_\oplus + d_{\text{Moon}}\frac{\theta_{\text{Sun}}}{2} \qquad \text{since } \theta_1 \approx \theta_2 \approx \theta_{\text{Sun}}/2$$ **2.0**

and

$$QB = QP - PB$$
$$= OE - PE\tan\theta_2$$
$$\approx R_\oplus - d_{\text{Moon}}\theta_2$$
$$\approx R_\oplus - d_{\text{Moon}}\frac{\theta_{\text{Sun}}}{2}$$ **2.0**

$$\therefore \theta_{\text{penumbra}} = 2\angle AOQ = 2\tan^{-1}\left(\frac{QA}{OQ}\right) \approx 2\frac{QA}{OQ} \qquad \text{since } QA \ll OQ$$

$$= 2\frac{R_\oplus + d_{\text{Moon}}\frac{\theta_{\text{Sun}}}{2}}{d_{\text{Moon}}} = \frac{2R_\oplus}{d_{\text{Moon}}} + \theta_{\text{Sun}}$$ **1.0**

and

$$\theta_{\text{umbra}} = 2\angle BOQ = 2\tan^{-1}\left(\frac{QB}{OQ}\right) \approx 2\frac{QB}{OQ}$$

$$= 2\frac{R_\oplus - d_{\text{Moon}}\frac{\theta_{\text{Sun}}}{2}}{d_{\text{Moon}}} = \frac{2R_\oplus}{d_{\text{Moon}}} - \theta_{\text{Sun}}$$ **1.0**

Subtracting,

$$\theta_{\text{penumbra}} - \theta_{\text{umbra}} = 2\theta_{\text{Sun}} \Rightarrow \theta_{\text{penumbra}} = \theta_{\text{umbra}} + 2\theta_{\text{Sun}}$$ **1.0**

We have,

$$\theta_{\text{umbra}} = 2.76\theta_{\text{Moon}} \qquad \text{and } \theta_{\text{Sun}} = 1915.0''$$

From the given data, θ_{Moon}= 2008.3″. **1.0**

Therefore,

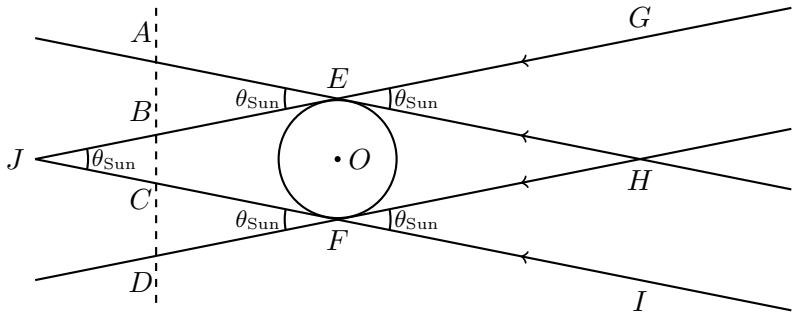
$$\theta_{\text{penumbra}} = 2.76\theta_{\text{Moon}} + 2\frac{1915.0}{2008.3}\theta_{\text{Moon}}$$

$$\boxed{\theta_{\text{penumbra}} = 4.67\theta_{\text{Moon}}}$$ **1.0**

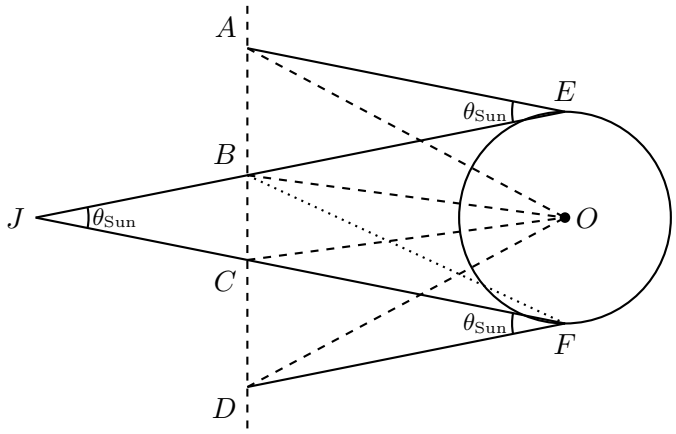
Acceptable range: $4.57\theta_{\text{Moon}}$ to $4.77\theta_{\text{Moon}}$.

Alternative solution:

In the figure below, rays HEA and IFC are coming from one edge of solar disk and rays HFD and GEB are coming from the opposite edge. The observer (O) is assumed to be at the centre of the Earth. The Moon travels along the path $ABCD$ during the course of eclipse.



Moon's path



Moon's path

From figure,

$$\measuredangle AEB = \measuredangle GEH = \measuredangle HFI = \measuredangle DFC = \measuredangle EJF = \theta_{\text{Sun}} \quad \textbf{3.0}$$

$$\theta_{\text{umbra}} = \measuredangle BOC = 2.76\theta_{\text{Moon}} \quad \textbf{1.0}$$

$$\theta_{\text{penumbra}} = \measuredangle AOD \quad \textbf{1.0}$$

$$\measuredangle AOD = \measuredangle AOB + \measuredangle BOC + \measuredangle COD$$

$$\measuredangle AOB = \measuredangle AEB \quad \textbf{1.0}$$

$$\measuredangle COD = \measuredangle CFD \quad \textbf{1.0}$$

$$\theta_{\text{penumbra}} \simeq \measuredangle AEB + \theta_{\text{umbra}} + \measuredangle CFD$$

$$= 2\theta_{\text{Sun}} + 2.76\theta_{\text{Moon}} = 2 \times 1915.0'' + 2.76 \times 2008.3''$$

$$\boxed{\theta_{\text{penumbra}} = 9372.9'' = 4.67\theta_{\text{Moon}}} \quad \textbf{2.0}$$

(D2.6) Let θ_{Earth} be angular size of the Earth as seen from the centre of the Moon. Calculate the angular size of the Moon, θ_{Moon}, as would be seen from the centre of the Earth on the eclipse day in terms of θ_{Earth}. **5**

Solution:

From the Moon,

$$\theta_{\text{Earth}} = \frac{2R_{\oplus}}{d_{\text{Moon}}} \quad \textbf{1.0}$$

From part (D2.5),

$$\theta_{\text{umbra}} + \theta_{\text{penumbra}} = 2\theta_{\text{Earth}} \quad \textbf{2.0}$$

$$\therefore \theta_{\text{Earth}} = \frac{\theta_{\text{umbra}} + \theta_{\text{penumbra}}}{2} = \frac{2.76 + 4.67}{2}\theta_{\text{Moon}} = 3.72\theta_{\text{Moon}}$$

$$\boxed{\theta_{\text{Moon}} = = 0.269\theta_{\text{Earth}}} \quad \textbf{2.0}$$

Alternative solution:

Let us say that the Moon is at position of B. Thus, angular size of Earth as seen from this position will be, (see figure in the previous part)

$$\theta_{\text{Earth}} = \measuredangle EBF = \measuredangle BFD \quad \textbf{2.0}$$

$$= \measuredangle BFC + \measuredangle CFD$$

$$\simeq \theta_{\text{umbra}} + \theta_{\text{Sun}} \quad \textbf{1.0}$$

The angular size of the Full Moon on 28 September as seen in the table is $2008.3''$.

$$\theta_{\text{Earth}} = 2.76 \times 2008.3'' + 1915.0'' = 7453.0''$$

$$\boxed{\theta_{\text{Moon}} = 0.269\theta_{\text{Earth}} = \frac{\theta_{\text{Earth}}}{3.72}} \quad \textbf{2.0}$$

(D2.7) Estimate the radius of the Moon, R_{Moon}, in km from the results above. **3**

Solution:

Thus, the radius of Moon will be,

$$R_{\text{Moon}} = \frac{R_\oplus}{3.72}$$

$$R_{\text{Moon}} = \frac{6371}{3.72} \quad \textbf{1.0}$$

$$\boxed{R_{\text{Moon}} \simeq 1713\,\text{km}} \quad \textbf{2.0}$$

Acceptable range: $\pm 20\,\text{km}$.

(D2.8) Estimate the shortest distance, r_{perigee}, and the farthest distance, r_{apogee}, to the Moon. **4**

Solution:

The shortest and longest distances will be,

$$r_{\text{perigee}} = \frac{2 \times 1713 \times 206265}{2008.3}$$

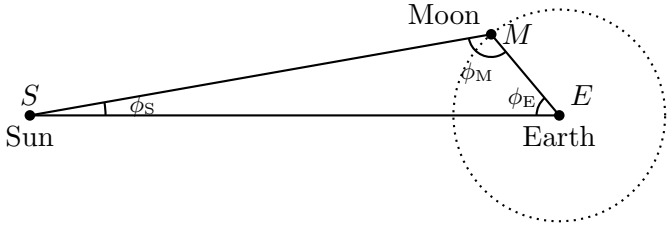
$$\boxed{r_{\text{perigee}} = 3.52 \times 10^5\,\text{km}} \quad \textbf{2.0}$$

$$r_{\text{apogee}} = \frac{2 \times 1713 \times 206265}{1763.7}$$

$$\boxed{r_{\text{apogee}} = 4.01 \times 10^5\,\text{km}} \quad \textbf{2.0}$$

(D2.9) Use appropriate data from September 10 to estimate the distance, d_{Sun}, to the Sun from the Earth. **10**

Solution:



Moon's orbit

On September 10, phase of Moon is 0.097 and elongation of Moon is 36.2°. Angular size of the Moon on this day is 1792.0″. Therefore, distance to Moon (from Earth) on September 10 is

$$d_{\text{Moon,10}} = \frac{2 \times 1713 \times 206265}{1792.0}$$
$$= 3.94 \times 10^5 \text{ km} \quad \textbf{2.0}$$

Let $\measuredangle EMS = \phi_M$

$\measuredangle ESM = \phi_S$

$\measuredangle SEM = \phi_E$

$$\therefore \phi_E = 36.2^\circ \quad \textbf{1.0}$$

$$\text{phase} = \frac{1 + \cos\phi_M}{2} \quad \textbf{2.0}$$

$$\therefore \phi_M = \cos^{-1}(2 \times \text{ phase } - 1)$$
$$= \cos^{-1}(2 \times 0.097 - 1) = \cos^{-1}(-0.806)$$
$$= 143.71^\circ \quad \textbf{2.0}$$

$$\phi_S = 180^\circ - \phi_E - \phi_M$$
$$= 180^\circ - 36.2^\circ - 143.71^\circ$$
$$= 0.09^\circ \quad \textbf{1.0}$$

Now using sine rule,

$$\frac{d_{\text{Sun}}}{d_{\text{Moon,10}}} = \frac{\sin\phi_M}{\sin\phi_S} \quad \textbf{2.0}$$

$$\therefore d_{\text{Sun}} = \frac{3.94 \times 10^8 \times \sin 143.71^\circ}{\sin 0.09^\circ}$$

$$\boxed{d_{\text{Sun}} = 1.48 \times 10^{11} \text{ m}} \quad \textbf{1.0}$$