Data Analysis: Instructions

- **Do not touch envelopes until the start of the examination.**
- The data analysis examination lasts for 3 hours and is worth a total of 125 marks.
- There are **Answer Sheets** for carrying out detailed work and **Working Sheets** for rough work, which are already marked with your student code and question number.
- ***Use only the answer sheets for a particular question for your answer. Please write only on the printed side of the sheet. Do not use the reverse side.*** If you have written something on any sheet which you do not want to be evaluated, cross it out.
- Use as many mathematical expressions as you think may help the evaluator to better understand your solutions. The evaluator may not understand your language. If it is necessary to explain something in words, please use short phrases (if possible in English).
- You are not allowed to leave your work desk without permission. If you need any assistance (malfunctioning calculator, need to visit a restroom, etc.), please draw the attention of the supervisor.
- The beginning and end of the examination will be indicated by the supervisor. The remaining time will be displayed on a clock.
- At the end of the examination you must stop writing immediately. Put everything back in the envelope and leave it on the table.
- Once all envelopes are collected, your student guide will escort you out of the examination room.
- A list of constants and useful relations are included in the envelope.

Data Analysis 2: 'Isolated black hole'

In 2022, two independent groups reported the discovery of an isolated black hole based on observations of the gravitational microlensing event OGLE-2011-BLG-0462. In this problem, we will analyze data from the Hubble Space Telescope to reproduce their findings.

Gravitational microlensing occurs when the light of a distant star (the 'source') is bent and magnified by the gravitational field of an intervening object (the 'lens'). The characteristic angular scale of gravitational microlensing events, called the angular Einstein radius $\theta_{\rm E}$, depends on the mass M and distance D_ℓ from the Earth to the lens:

$$\theta_{\rm E} = \sqrt{\frac{4GM}{c^2}\frac{D_s - D_\ell}{D_s D_\ell}},$$

where D_s is the distance to the source star. For typical microlensing events observed in the Milky Way, the source stars are in the Galactic bulge, near the Galactic center, so $D_s \approx 8\,\mathrm{kpc}$.

(a) Calculate the angular Einstein radius in milliarcseconds (mas) for an example lens of $1\,M_\odot$ located at a distance of 1 kpc. (2 points)

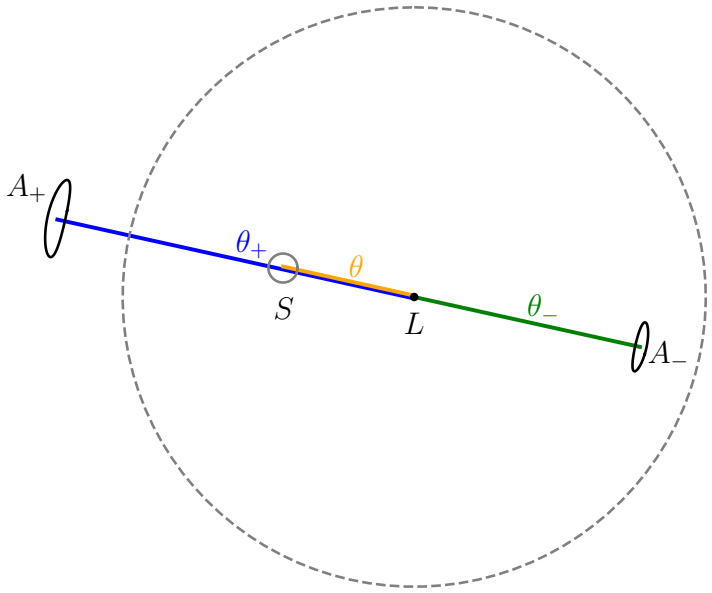
Suppose that at time t the lens and the source are separated by an angle $\theta \equiv u(t)\theta_{\rm E}$ on the sky. Two images of the source are created on a line through the positions of the source and the lens, at angular distances θ_+ and θ_- from the lens given by:

$$\theta_\pm = \frac{1}{2}\left(u \pm \sqrt{u^2+4}\right)\theta_{\rm E}.$$

These two images are magnified, relative to the unlensed brightness of the source. The absolute magnification of the images is:

$$A_\pm = \frac{1}{2}\left(\frac{u^2+2}{u\sqrt{u^2+4}} \pm 1\right).$$

The image below shows the geometry of the event. The position of the lens is marked as L, the unlensed position of the source is marked as S, while A_+ and A_- mark the positions of the two images of the source. The dashed circle has a radius of one Einstein radius.

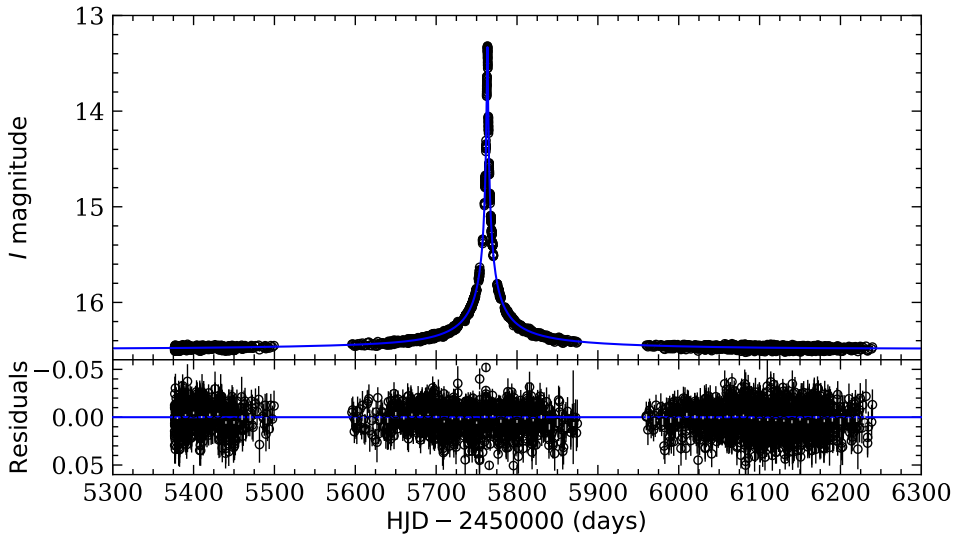


(b) Current telescopes cannot normally resolve this pair of images, but only measure the position of the image centroid, i.e. the brightness-weighted mean of the positions of the two images. Derive an expression for the angular separation θ_c of the image centroid relative to the lens as a function of u and $\theta_{\rm E}$. (8 points)

(c) Derive an expression for the source deflection $\Delta\theta$, i.e. the difference between the location of the centroid and the unlensed position of the source, as a function of u and $\theta_{\rm E}$. What is the source deflection when the lens and the source are nearly perfectly aligned ($u \approx 0$)? (4 points)

The source and lens are moving relative to each other in the sky. Thus, both the total magnification of the images and the position of the centroid changes with time, resulting in observable photometric and astrometric microlensing effects. For now, we assume that the source-lens relative motion is rectilinear.

The plot below shows the light curve of the gravitational microlensing event OGLE-2011-BLG-0462, discovered by the OGLE sky survey led by astronomers from the University of Warsaw. The solid line shows the best-fitting light curve model. The Einstein timescale of the event, i.e. the time needed for the source to move by one angular Einstein radius relative to the lens, was $t_{\rm E} = 247\,$days. The event peaked on 21 July 2011 (HJD = 2455763). The minimal separation between the lens and the source was $u_0 \approx 0$.



The table below shows the measured positions of the source star against the background objects in the East and North directions based on images from the Hubble Space Telescope.

HJD	E position (mas)	N position (mas)
2455765.2	2.58 ± 0.13	7.29 ± 0.16
2455865.7	2.32 ± 0.12	5.44 ± 0.24
2456179.7	0.46 ± 0.14	1.62 ± 0.08
2456195.8	0.88 ± 0.36	1.56 ± 0.77
2456426.2	-1.02 ± 0.21	-0.94 ± 0.12
2456587.7	-2.04 ± 0.07	-1.88 ± 0.40
2456956.6	-4.54 ± 0.25	-5.16 ± 0.29
2457995.2	-11.14 ± 0.12	-15.14 ± 0.17

(d) Plot the measured positions of the source star against the background objects in the East and North directions as a function of time. (10 points)

(e) The observed motion of the source star is the sum of two effects: rectilinear proper motion of the source and astrometric microlensing effects. Calculate the proper motion (in mas/year) of the source and its uncertainty in the East and North directions. (8 points)

(f) After subtracting the effects of proper motion from the data, calculate and plot the total resultant astrometric deflection as a function of u. Neglect the uncertainty of the proper motion determination. (20 points)

(g) Analyse the data to determine the angular Einstein radius $\theta_{\rm E}$ of the event and its uncertainty. (Hint: it may be helpful to linearise the expression for $\Delta\theta$).

(16 points)

(h) For long-timescale events such as OGLE-2011-BLG-0462, the rectilinear approximation of the relative lens-source proper motion is not strictly true and the orbital motion of the Earth has to be taken into account. This allows measurement of a dimensionless quantity called the microlensing parallax, defined as $\pi_{\rm E} = (\pi_l - \pi_s)/\theta_{\rm E}$, where π_l and π_s are parallaxes of the lens and the source, respectively.

For this event $\pi_{\rm E} = 0.095 \pm 0.009$. Rearrange the expression for $\theta_{\rm E}$ given earlier to calculate the mass of the lens in solar masses and its uncertainty.

(7 points)

(Total: 75 points)

Solution

(a)

$$\theta_{\rm E} = \sqrt{\frac{4GM}{c^2}\frac{D_s - D_\ell}{D_s D_\ell}} = \sqrt{\frac{4GM}{{\rm au}c^2}\left(\frac{\rm au}{D_\ell} - \frac{\rm au}{D_s}\right)} = 2.7\,{\rm mas}$$

The angular resolution of modern large ($D \approx 10\,{\rm m}$) optical ($\lambda = 550\,{\rm nm}$) telescopes is $\theta_0 = 1.22\lambda/D \approx 14\,{\rm mas}$. Thus, $\theta_0 \gg \theta_{\rm E}$, so the images created during microlensing events cannot be resolved by these telescopes.

(2 points)

(b)

$$\begin{aligned}\theta_c &= \frac{\theta_+ A_+ + \theta_- A_-}{A_+ + A_-} = \frac{\frac{1}{4}\left(u + \sqrt{u^2+4}\right)\left(\frac{u^2+2}{u\sqrt{u^2+4}} + 1\right) + \frac{1}{4}\left(u - \sqrt{u^2+4}\right)\left(\frac{u^2+2}{u\sqrt{u^2+4}} - 1\right)}{\frac{u^2+2}{u\sqrt{u^2+4}}}\theta_{\rm E} \\ &= \frac{\left(u + \sqrt{u^2+4}\right)\left(u^2 + 2 + u\sqrt{u^2+4}\right) + \left(u - \sqrt{u^2+4}\right)\left(u^2 + 2 - u\sqrt{u^2+4}\right)}{4(u^2+2)}\theta_{\rm E} \\ &= \frac{2u(u^2+2) + 2u(u^2+4)}{4(u^2+2)}\theta_{\rm E} = \frac{2u(2u^2+6)}{4(u^2+2)}\theta_{\rm E} = \frac{u(u^2+3)}{u^2+2}\theta_{\rm E}\end{aligned}$$

(8 points)

(c)

$$\Delta\theta = \theta_c - \theta = \frac{u(u^2+3)}{u^2+2}\theta_{\rm E} - u\theta_{\rm E} = \frac{u(u^2+3) - u(u^2+2)}{u^2+2}\theta_{\rm E} = \frac{u}{u^2+2}\theta_{\rm E}$$

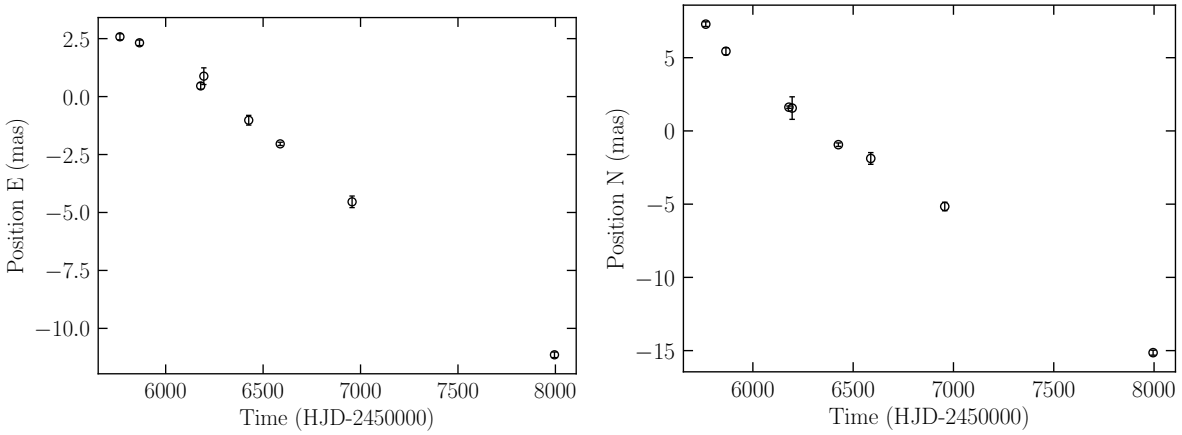
(3 points)

$$\Delta\theta(u = 0) = 0$$

so there is no deflection when the lens and the source are nearly perfectly aligned.

(1 points)

(d)



(10 points, 5 points for each graph)

(e) I will use the fact that the first epoch of astrometric observations was taken close to the peak of the light curve (that is, $u_1 \approx 0$, that is, almost no astrometric deflection). Similarly, astrometric deflection is close to zero for the last epoch. Thus,

$$\mu_E \approx \frac{x_{E,8} - x_{E,1}}{t_8 - t_1} = -2.247 \pm 0.029\,\mathrm{mas/yr}$$
$$\mu_N \approx \frac{x_{N,8} - x_{N,1}}{t_8 - t_1} = -3.674 \pm 0.038\,\mathrm{mas/yr}$$

(8 points)

No points should be given if a student does not recognize that the deflection is zero during the first epoch, e.g., they try fitting a straight line to all data points.

(f) I fitted a straight line joining the first and last epoch data and then subtracted it from astrometric measurements. This is because the observed path of the source on the sky is the sum of two effects: the rectilinear proper motion of the source and the astrometric deflection:

$$x(E) = x_1^E + (t - t_1)\mu_\mathrm{E} + \Delta\theta(E)$$
$$x(N) = x_1^N + (t - t_1)\mu_\mathrm{N} + \Delta\theta(N),$$

where x_1^E is the East position of the source during the first epoch, x_1^N is the North position of the source during the first epoch, t_1 is the time of the first observation, and $\Delta\theta(E)$ and $\Delta\theta(N)$ is the astrometric deflection due to microlensing in East and North direction, respectively.

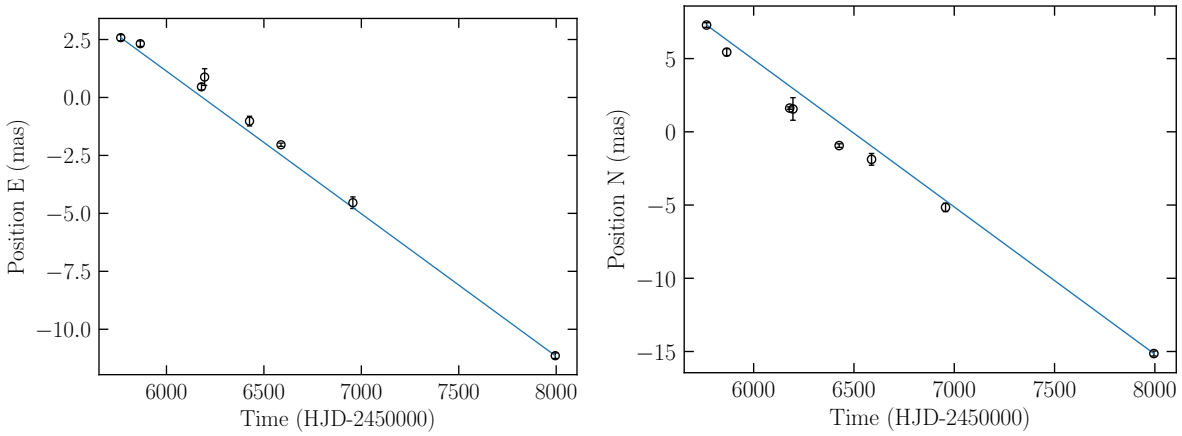


Figure 7: The blue lines join the first and the last epoch data points. The source would move along these lines if there wasnt't any black hole in front of it. However, the position of the source that was observed is deflected due to microlensing effects by the black hole.

Thus, the astrometric deflection during ith epoch is:

$$\Delta\theta(E)_i = x_i^E - x_1^E - (t_i - t_1)\mu_\mathrm{E}$$
$$\Delta\theta(N)_i = x_i^N - x_1^N - (t_i - t_1)\mu_\mathrm{N}$$

and the total deflection is:

$$\Delta\theta_i = \sqrt{\Delta\theta(E)_i^2 + \Delta\theta(N)_i^2}.$$

I will also use the fact that $u_0 \approx 0$, so $u = (t - t_0)/t_E$, where $t_0 = 2455763$ is the peak time. Results of my calculations are shown in the table below.

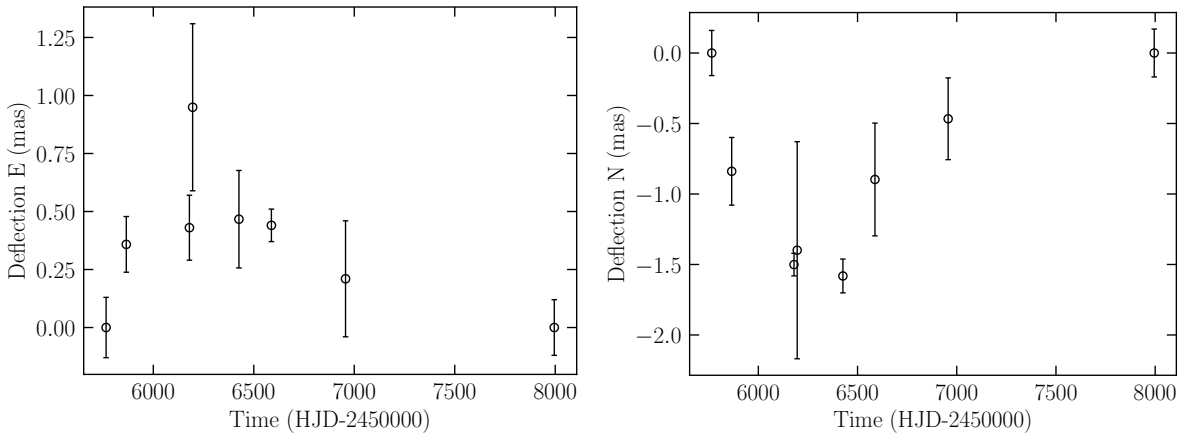


Figure 8: These figures show the astrometric deflection in East and North directions induced by the black hole. (Students are not required to make these plots.)

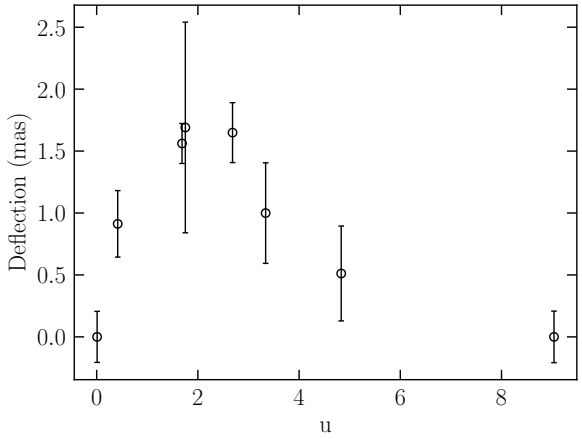


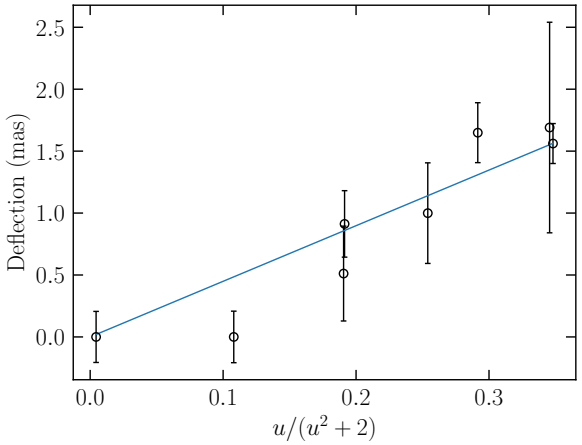
Figure 9: Total astrometric deflection as a function of u. Students are asked to make this plot in part (f).

Epoch	u	$\Delta\theta$ (E,mas)	$\Delta\theta$ (N,mas)	$\Delta\theta$ (mas)
1	0.01	0.00 ± 0.13	-0.00 ± 0.16	0.00 ± 0.21
2	0.42	0.36 ± 0.12	-0.84 ± 0.24	0.91 ± 0.27
3	1.69	0.43 ± 0.14	-1.50 ± 0.08	1.56 ± 0.16
4	1.75	0.95 ± 0.36	-1.40 ± 0.77	1.69 ± 0.85
5	2.69	0.47 ± 0.21	-1.58 ± 0.12	1.65 ± 0.24
6	3.34	0.44 ± 0.07	-0.90 ± 0.40	1.00 ± 0.41
7	4.83	0.21 ± 0.25	-0.47 ± 0.29	0.51 ± 0.38
8	9.04	0.00 ± 0.12	-0.00 ± 0.17	0.00 ± 0.21

(20 points)

(g) We would like to fit the function $\Delta\theta = \frac{u}{u^2+2}\theta_{\rm E}$ to the data. Thus, my "new" independent variable would be $x' = u/(u^2+2)$. Now, I would like to fit the function $y' = \theta_{\rm E} x'$, where $y' = \Delta\theta$. Thus

$$\theta_{\rm E} = \frac{\sum y'_i x'_i/\sigma_i^2}{\sum x_i'^2/\sigma_i^2} \pm \frac{1}{\sqrt{\sum x_i'^2/\sigma_i^2}} = 4.5 \pm 0.4\,{\rm mas}$$



Alternative solution:

The gravitational deflection is described by the formula $\Delta\theta = \frac{u}{u^2+2}\theta_{\rm E}$. It can be demonstrated that this function reaches a local maximum for $u = \sqrt{2}$ with $\Delta\theta_{\rm max} = \frac{\sqrt{2}}{4}\theta_{\rm E} = 0.354\theta_{\rm E}$. From the data, we can estimate the maximum deflection of $\Delta\theta_{\rm max} = 1.59 \pm 0.14\,$mas. Thus, $\theta_{\rm E} = 4.5 \pm 0.4\,$mas.

(16 points)

(h)

Let $\pi_{\rm rel} = \pi_l - \pi_s$ be the relative lens–source parallax. From the definition of the angular Einstein radius we have

$$\theta_{\rm E} = \sqrt{\frac{4GM}{c^2}\frac{D_s - D_l}{D_s D_l}} = \sqrt{\frac{4GM}{c^2}\left(\frac{1}{D_l} - \frac{1}{D_s}\right)} = \sqrt{\frac{4GM}{c^2{\rm au}}\left(\frac{\rm au}{D_l} - \frac{\rm au}{D_s}\right)} = \sqrt{\frac{4GM\pi_{\rm rel}}{c^2{\rm au}}}.$$

From the definition of the microlensing parallax $\pi_{\rm rel} = \pi_{\rm E}\theta_E$, so this equation becomes:

$$\theta_{\rm E} = \sqrt{\frac{4GM\pi_{\rm E}\theta_{\rm E}}{c^2{\rm au}}},$$

and hence

$$M = \frac{\theta_{\rm E}c^2 AU}{4G\pi_{\rm E}} = 5.8 \pm 0.8 M_\odot.$$

The uncerainty on M can be determined from the relation:

$$\frac{\Delta M}{M} = \sqrt{\left(\frac{\Delta\theta_E}{\theta_{\rm E}}\right)^2 + \left(\frac{\Delta\pi_{\rm E}}{\pi_{\rm E}}\right)^2}$$

(7 points)

Grading

a	Correct value of θ_{E}	2
b	Correct formula	8
c	Correct formula	4
d	Correct plots (5 pts for each plot)	10
e	Correct μ_E within 3σ	3
	Correct μ_E within 5σ	1
	Incorrect μ_E ($> 5\sigma$)	0
	Uncertainty on μ_E	1
	Correct μ_N within 3σ	3
	Correct μ_N within 5σ	1
	Incorrect μ_N ($> 5\sigma$)	0
	Uncertainty on μ_N	1
	No points should be given if the student does not recognize that the deflection is zero during the first epoch	
f	Calculation of the impact parameter u for all epochs	4
	Calculation of the total deflection	6
	Calculation of the uncertainties	5
	Graph	5
	Grant full points if results are correct for 7 or 8 epochs	
	Grant 60% points if results are correct for 5 or 6 epochs	
	Grant 0 points if results are correct for 4 or less than 4 epochs	
g	Correct result with an estimate of the uncertainty	16
	Correct results without the estimate of the uncerainty	8
h	Correct result	7
	TOTAL	75

Grading of the graphs:

- Students can get 5 pts for each correct graphs
- data points with error bars 3 pts
- axis labels with units, tick labels - 1 pts
- graph is clear, fills the entire area - 1 pts

Grant full points if the graph shows correct data for 7 or 8 epochs. Grant 60% points if the graph shows correct data for 5 or 6 epochs. Grant 0 points if the graph shows 4 or less than 4 epochs.