Solution

A1. The length of $\vec{M}$ squared is equal to

$$M^2 = \vec{M} \cdot \vec{M},$$

thus, its time derivative is

$$\frac{\partial}{\partial t} M^2 = 2\vec{M} \cdot \frac{\partial \vec{M}}{\partial t}.$$

For any vectors $\vec{a}, \vec{b}$, their vector product $\vec{a} \times \vec{b}$ is orthogonal to the vectors. In particular, vectors $\vec{M} \times \vec{B}_{\text{eff}}$ and $\vec{M} \times \frac{\partial \vec{M}}{\partial t}$ are orthogonal to $\vec{M}$, so according to LLG equation $\vec{M}$ and $\frac{\partial \vec{M}}{\partial t}$ are perpendicular, therefore

$$\vec{M} \cdot \frac{\partial \vec{M}}{\partial t} = 0, \quad \frac{\partial}{\partial t} M^2 = 0,$$

and the magnitude of $\vec{M}$ is constant.

$$\boxed{\frac{\partial}{\partial t} M^2 = 2\vec{M} \cdot \frac{\partial \vec{M}}{\partial t} = 0}$$

A2. Consider the z-component of the equation

$$\frac{\mathrm{d}\vec{m}}{\mathrm{d}t} = -\frac{\gamma}{1+\alpha^2} \vec{m} \times \vec{B}_{\text{eff}} - \frac{\alpha\gamma}{1+\alpha^2} \left((\vec{m} \cdot \vec{B}_{\text{eff}}) \, \vec{m} - \vec{B}_{\text{eff}} \right).$$

The vector product $(\vec{m} \times \vec{B}_{\text{eff}})$ is perpendicular to z axis, so only the second term contributes to the equation. Taking into account $\vec{m} \cdot \vec{B}_{\text{eff}} = m_z B_0$ we get

$$\dot{m}_z = \frac{\alpha\gamma}{1+\alpha^2} B_0 (1 - m_z^2).$$

Separating variables

$$\frac{\mathrm{d}m_z}{1 - m_z^2} = \frac{\alpha\gamma}{1+\alpha^2} B_0 \, \mathrm{d}t$$

and integrating the resulting expression from 0 to t we obtain

$$\frac{1}{2} \ln \left(\frac{1 + m_z(t)}{1 - m_z(t)} \right) \Bigg|_0^t = \frac{\alpha\gamma}{1+\alpha^2} B_0 t.$$

Substituting initial condition $m_z(0) = 0$,

$$\frac{1}{2} \ln \left(\frac{1 + m_z(t)}{1 - m_z(t)} \right) = \frac{\alpha\gamma}{1+\alpha^2} B_0 t,$$

we find the answer

$$\boxed{m_z(t) = \frac{\exp\left(\frac{\alpha\gamma B_0 t}{1+\alpha^2}\right) - \exp\left(-\frac{\alpha\gamma B_0 t}{1+\alpha^2}\right)}{\exp\left(\frac{\alpha\gamma B_0 t}{1+\alpha^2}\right) + \exp\left(-\frac{\alpha\gamma B_0 t}{1+\alpha^2}\right)} = \tanh\left(\frac{\alpha\gamma B_0 t}{1+\alpha^2}\right)}$$

A3. The first solution

Denote by $\vec{m}_{xy}$ the projection of the vector $\vec{m}$ onto the xy-plane. The vector $\vec{B}_{\text{eff}}$ is directed along the z axis, so that its projection onto the xy-plane is zero. Next, $\vec{m} \cdot \vec{B}_{\text{eff}} = m_z B_0$ and

$$\vec{m} \times \vec{B}_{\text{eff}} = (\vec{m}_{xy} + m_z \vec{e}_z) \times \vec{B}_{\text{eff}} = \vec{m}_{xy} \times \vec{B}_{\text{eff}}, \quad (\vec{e}_z \times \vec{B}_{\text{eff}} = 0),$$

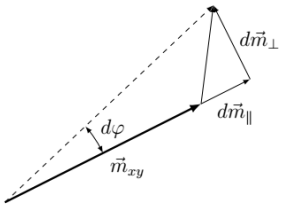
so that the projection of the LLG equation onto the xy-plane takes the form

$$\frac{\mathrm{d}\vec{m}_{xy}}{\mathrm{d}t} = -\frac{\gamma}{1+\alpha^2}\vec{m}_{xy} \times \vec{B}_{\text{eff}} - \frac{\alpha\gamma}{1+\alpha^2} m_z B_0 \, \vec{m}_{xy}.$$

Here the first term (with magnitude $\gamma B_0 m_{xy}/(1+\alpha^2)$) is orthogonal to $\vec{m}_{xy}$ and hence leads to the rotation of $\vec{m}_{xy}$. The second term is parallel to $\vec{m}_{xy}$ and leads to the length change.

During the time period $\mathrm{d}t$, a perpendicular change component is

$$\mathrm{d}m_\perp = \frac{\gamma B_0 m_{xy}}{1+\alpha^2}\mathrm{d}t,$$



so that the rotation angle is $\mathrm{d}\varphi = \mathrm{d}m_\perp / m_{xy}$, and the angular velocity is

$$\Omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\gamma B_0}{1+\alpha^2}.$$

Another way to obtain Ω is to notice that if some vector rotates with an angular velocity $\vec{\Omega}$, then its time derivative is equal to

$$\frac{\mathrm{d}\vec{m}_{xy}}{\mathrm{d}t} = \vec{\Omega} \times \vec{m}_{xy}.$$

Comparing with the first term in the LLG equation, we immediately get

$$\vec{\Omega} = \frac{\gamma \vec{B}_{\text{eff}}}{1+\alpha^2} = \frac{\gamma B_0}{1+\alpha^2}\vec{e}_z.$$

So the answer is

$$\boxed{\Omega = \frac{\gamma B_0}{1+\alpha^2}}$$

A3. The second solution

Project the LLG equation onto the x and y axis:

$$\begin{cases} \dot{m}_x = -\dfrac{\gamma B_0}{1+\alpha^2} m_y - \dfrac{\alpha\gamma B_0 m_z}{1+\alpha^2} m_x, \\ \dot{m}_y = \dfrac{\gamma B_0}{1+\alpha^2} m_x - \dfrac{\alpha\gamma B_0 m_z}{1+\alpha^2} m_y. \end{cases}$$

Introduce a new complex variable $\hat{m} = m_x + i m_y$ and take a sum of two equations above, where the second equation is multiplied by i firstly:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_x + i m_y) = -\frac{\gamma B_0}{1+\alpha^2}(-m_y + i m_x) - \frac{\alpha\gamma m_z B_0}{1+\alpha^2}(m_x + i m_y).$$

The resulting equation can be rewritten as

$$\frac{\mathrm{d}\hat{m}}{\mathrm{d}t} = i\frac{\gamma B_0}{1+\alpha^2}\hat{m} - \frac{\alpha\gamma B_0 m_z}{1+\alpha^2}\hat{m}.$$

Separate variables and get

$$\int \frac{\mathrm{d}\hat{m}}{\hat{m}} = \int \mathrm{d}t \left(\frac{i\gamma B_0}{1+\alpha^2} - \frac{\alpha\gamma B_0}{1+\alpha^2} m_z \right).$$

Due to the initial condition $\hat{m}(0) = m_x(0) + i m_y(0) = 1$ we have

$$\ln \hat{m} = \frac{i\gamma B_0 t}{1+\alpha^2} - \frac{\alpha\gamma B_0}{1+\alpha^2} \int_0^t \mathrm{d}t\, m_z(t), \quad \hat{m} = A(t) e^{i\varphi(t)},$$

where the expression

$$A(t) = \exp\left(-\frac{\alpha\gamma B_0}{1+\alpha^2} \int_0^t m_z(t)\, \mathrm{d}t \right)$$

is the magnitude of $\vec{m}_{xy}$, and the expression

$$\varphi = \frac{\gamma B_0 t}{1+\alpha^2}$$

corresponds to an angle of rotation of $\vec{m}_{xy}$ since

$$m_x(t) = A(t)\cos\varphi(t), \quad m_y(t) = A(t)\sin\varphi(t).$$

The angle φ is a linear function in t, so that the angular velocity is constant, and its value coincides with the one obtained earlier. Using the explicit formula for $m_z(t)$ one can find the magnitude $A(t)$. Indeed,

$$\ln A(t) = -\frac{\alpha\gamma B_0}{1+\alpha^2} \int_0^t m_z(t)\, \mathrm{d}t = -\int_0^t \frac{\alpha\gamma B_0 \mathrm{d}t}{1+\alpha^2} \tanh \frac{\alpha\gamma B_0 t}{1+\alpha^2} = -\ln\cosh \frac{\alpha\gamma B_0 t}{1+\alpha^2},$$

what implies

$$A(t) = \frac{1}{\cosh \dfrac{\alpha\gamma B_0 t}{1+\alpha^2}},$$

so that $m_z^2 + m_{xy}^2 = 1$, i.e. the magnitude of $\vec{m}$ is constant and equals 1.

Here we use the formula

$$\int \tanh x\, \mathrm{d}x = \int \frac{\sinh x}{\cosh x} \mathrm{d}x = \int \frac{\mathrm{d}(\cosh x)}{\cosh x} = \ln\cosh x + C,$$

and the relation

$$\tanh^2 x + \frac{1}{\cosh^2 x} = 1.$$

One can avoid the use of complex numbers and find the solution in the form

$$m_x = A(t)\cos\varphi(t), \quad m_y(t) = A(t)\sin\varphi(t).$$

Substitute these expressions to the equations for $\dot{m}_x, \dot{m}_y$ and get

$$\frac{\mathrm{d}A}{\mathrm{d}t}\cos\varphi - A\sin\varphi\frac{\mathrm{d}\varphi}{\mathrm{d}t} = -\frac{\gamma B_0}{1+\alpha^2} A\sin\varphi - \frac{\alpha\gamma B_0 m_z}{1+\alpha^2} A\cos\varphi;$$

$$\frac{\mathrm{d}A}{\mathrm{d}t}\sin\varphi + A\cos\varphi\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\gamma B_0}{1+\alpha^2} A\cos\varphi - \frac{\alpha\gamma B_0 m_z}{1+\alpha^2} A\sin\varphi.$$

From these equations one can get the time derivatives of the magnitude and the angle:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\gamma B_0}{1+\alpha^2}, \quad \frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{\alpha\gamma B_0 m_z}{1+\alpha^2} A.$$

Note that these relations are nothing but the equalities of the coefficients at $\cos\varphi$ and $\sin\varphi$ in the initial system of equations. The first equation gives a former expression for an angular velocity, and the second one — for the magnitude derivative.

$$\boxed{\Omega = \frac{\gamma B_0}{1+\alpha^2}}$$

B1. By definition

$$\vec{B}_{\text{eff},i} = -\frac{1}{a^3}\frac{\partial W(\vec{M}_i)}{\partial \vec{M}_i},$$

where W is the energy of the system. It consists of contributions given in the task:

$$W = \sum_i a^3 \left(-\frac{K M_{i,z}^2}{2M_0^2} + \frac{K_1 M_{i,y}^2}{2M_0^2}\right) - \sum_i A\vec{M}_i\vec{M}_{i+1}\frac{a}{M_0^2} - \sum_i a^3 \vec{B}_0\vec{M}_i.$$

Here the first term represents anisotropy energy, only one term in it depends on a magnetization of every cell, so the corresponding magnetic field has the form

$$\vec{B}_{an,i} = \frac{\partial}{\partial M_{z,i}}\left(\frac{K M_{i,z}^2}{2M_0^2}\right)\vec{e}_z - \frac{\partial}{\partial M_{y,i}}\left(\frac{K_1 M_{i,y}^2}{2M_0^2}\right)\vec{e}_y = \frac{K M_{i,z}}{M_0^2}\vec{e}_z - \frac{k_1 K M_{i,y}}{M_0^2}\vec{e}_y = \frac{K}{M_0}\left(m_{i,z}\vec{e}_z - k_1 m_{i,y}\vec{e}_y\right).$$

Comparing with the formula from the task we can find

$$B_{\text{an}} = \frac{K}{M_0}.$$

The second term in W represents the exchange energy, two terms in it depend on $\vec{M}_i$

$$-\frac{Aa}{M_0^2}\vec{M}_i(\vec{M}_{i-1} + \vec{M}_{i+1}),$$

so the effective magnetic field is

$$\vec{B}_{\text{ex}} = -\frac{1}{a^3}\frac{\partial}{\partial \vec{M}_i}\left(-\frac{Aa}{M_0^2}\vec{M}_i(\vec{M}_{i-1} + \vec{M}_{i+1})\right) = \frac{A}{a^2 M_0^2}(\vec{M}_{i-1} + \vec{M}_{i+1}) = \frac{A}{a^2 M_0}(\vec{m}_{i-1} + \vec{m}_{i+1}).$$

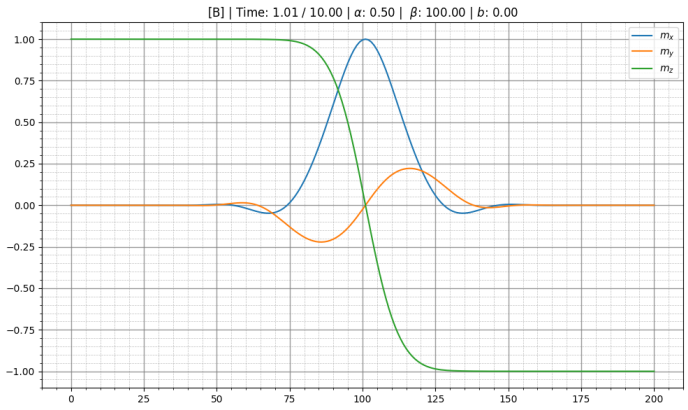
Therefore

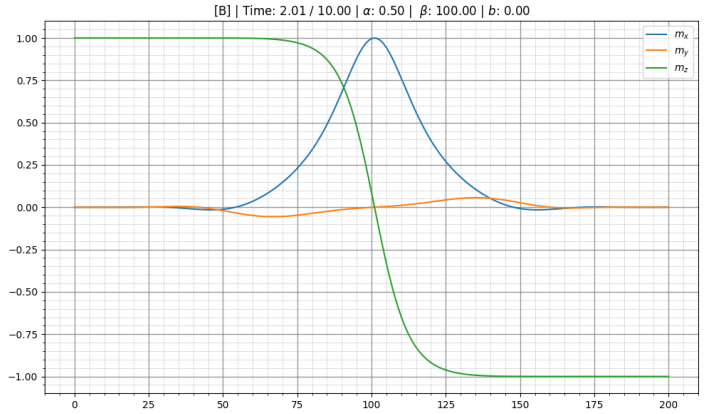
$$B_{\text{ex}} = \frac{A}{a^2 M_0}.$$

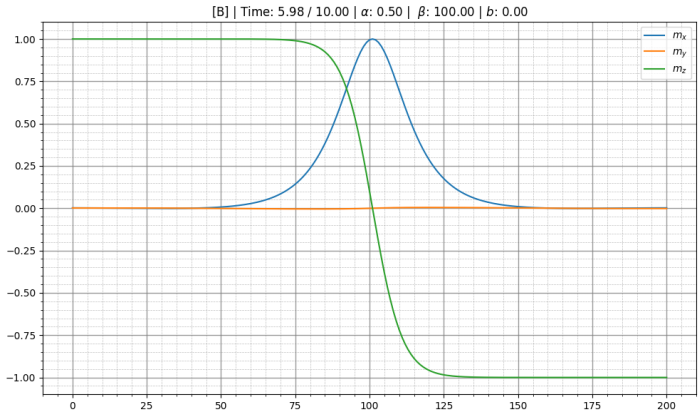
Differentiating the last term, we get a contribution to the effective magnetic field from the external magnetic field $\vec{B}_0$ as it is stated in the task.

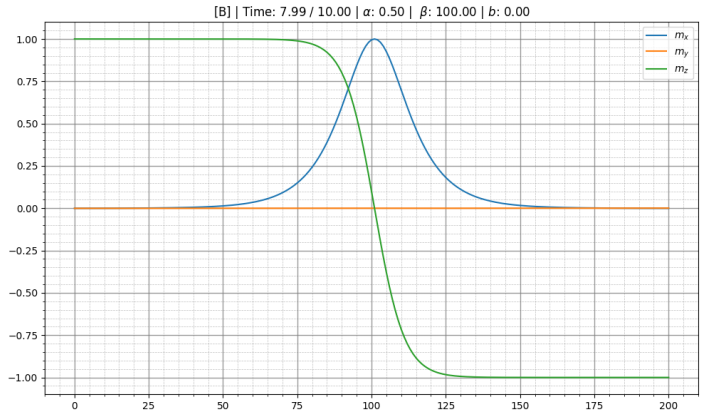
$$\boxed{B_{\text{an}} = \frac{K}{M_0} \approx 0.83\ \text{T}; \qquad B_{\text{ex}} = \frac{A}{M_0 a^2} \approx 83.33\ \text{T}; \qquad \omega \approx 1.47 \cdot 10^{11}\ \text{s}^{-1}.}$$

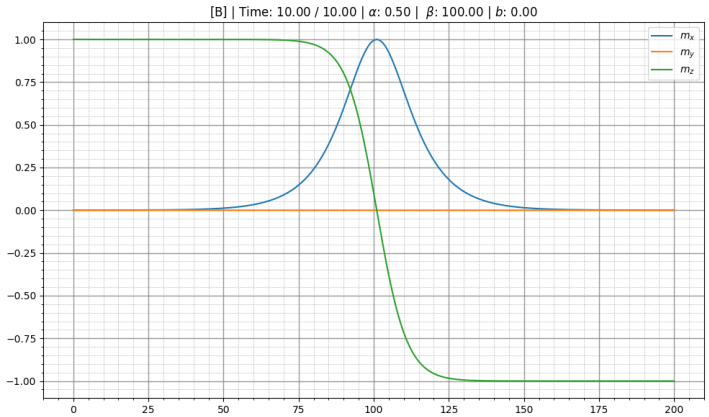
B2. Let us run the program for $\alpha = 0.5$, $\beta = 100$, $b = 0$, $T = 10$. The pictures show graphs of m_x, m_y, m_z for different t in range $[0, 10]$. Note that the magnetization distribution on the last two pictures is approximately the same, so we can consider it stationary. In particular, one of the features of the stationary state regime is that $m_y(x) = 0$ for all x.



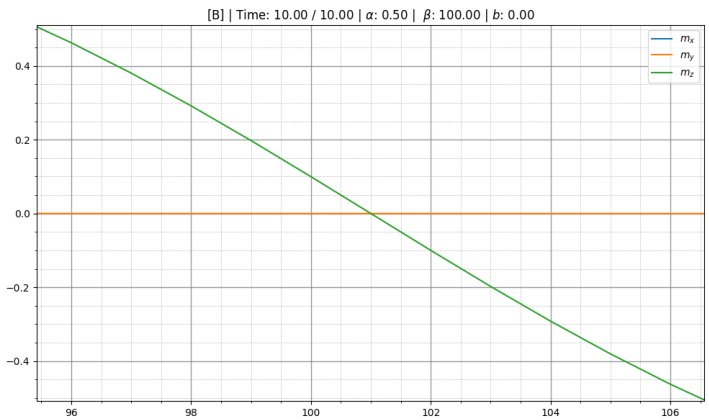








To increase an accuracy of d measurement let us zoom in the part of the plot $m_z(x)$ for $t = 10$ in such a way that the side points correspond to $m_z = \pm 0.5$. The x-coordinates of the points with $m_z = 0.5$ and $m_z = -0.5$ are equal to $x_{\text{left}} = 95.5$ and $x_{\text{right}} = 106.5$ respectively, so that $d = x_{\text{right}} - x_{\text{left}} = 11.0$.



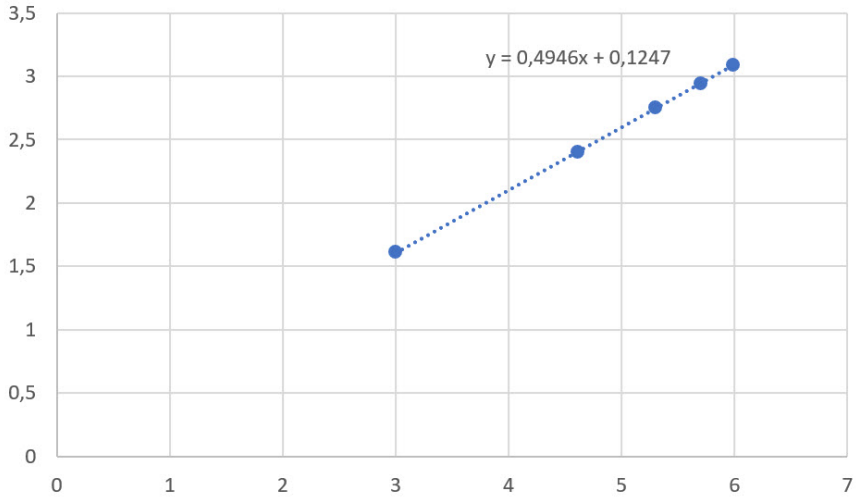
$$\boxed{d = 11.0}$$

B3. In part B2 we have seen that for $\alpha = 0.5$, $\beta = 100$, $b = 0$ and $T = 10$, the steady state regime occurs in the end. Thus, in the present part we will use the same values of the parameters α and T, and different $\beta \in [20, 400]$. We will find the width d in the way described in the solution of B2, for $t = 10$. To ensure that the system is indeed in stationary state, we check that $m_y \equiv 0$ at $t = 10$ on the graph. Denote by x_{left}, (resp. x_{right}) the coordinates of the left (resp. right) borders of the domain wall. The width of the wall can be computed using the formula $d = x_{\text{right}} - x_{\text{left}}$.

β	x_{left}	x_{right}	d
20	98.5	103.5	5.0
100	95.5	106.5	11.0
200	93.2	108.8	15.6
300	91.5	110.5	19.0
400	90.0	112.0	22.0

B4. Let $d(\beta) = \gamma\beta^c$, where γ is a constant coefficient. Then $\ln d(\beta) = \ln\gamma + c\ln\beta$, so that the dependence $\ln d\ (\ln\beta)$ is linear, and the slope is equal to c. Calculate $\log d$ and $\log\beta$ for the points obtained in B3, and then draw a graph in Excel using the resulting data.

$\ln\beta$	$\ln d$
3.00	1.61
4.61	2.40
5.30	2.75
5.70	2.94
5.99	3.09



The plot $\ln d\ (\ln\beta)$.

$$\boxed{c = 0.495}$$

C1. Set some $b \neq 0$ and run the program. We see that the domain wall moves to the right.

$$\boxed{\text{To the right.}}$$

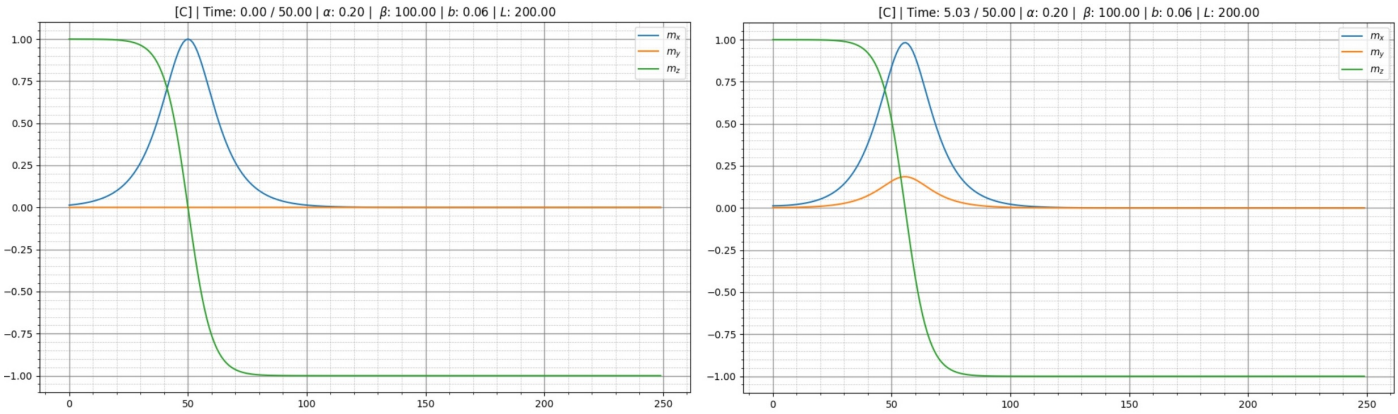
C2. After the start of the program, the transient process begins. For small enough values of b the system settles into uniform motion, and the angle of magnetic fields rotation φ stops changing. The evolution time should be large enough so that the motion becomes uniform. The length L should be such that the domain wall does not reach the right side during the evolution time. On the other hand, with the growth of T and L, the runtime increases, so that one should find optimal parameters.

Since just after the start of the program the uniform motion is not established yet, the velocity can not be computed using the formula $v = \frac{x-x_0}{t}$, where x_0 is the initial position of the center of the domain wall. The measurements should be taken in a steady state regime, so that one needs to determine the duration of the transient process. To do this one need to measure the dependence $x(t)$ for some b, and estimate the typical transient time. Another way to get correct measurements is to take them only when the shape of m_y stops changing over time (one can see the change of m_y only after zooming the plot). In a steady state regime, due to a high accuracy of measurements, it is enough to compute the velocity of the center of the wall by the formula $v = \frac{x_2-x_1}{t_2-t_1}$, i.e. using only two points.

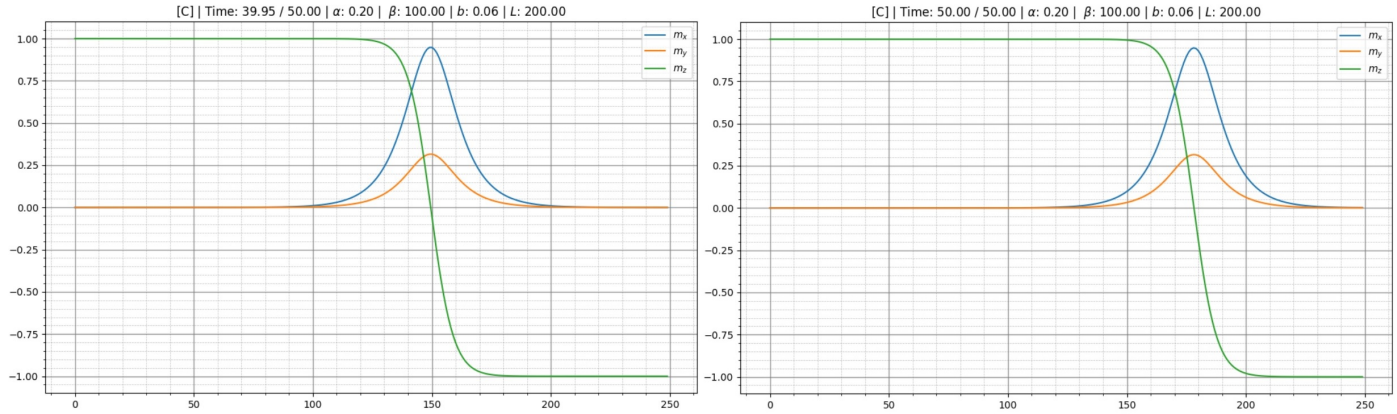
In the center of the domain wall we have $m_z = 0$, and both $m_x = m_{x0}$, $m_y = m_{y0}$ are maximal. (One can see from the plots that the coordinates of the maxima of m_x and m_y and the coordinate x such that $m_z(x) = 0$, almost coincide). Therefore, we can find the angle φ using either of the equivalent relations $m_{x0} = \cos\varphi,\ m_{y0} = \sin\varphi$, but only when the uniform motion is already established. Thus,

$$\varphi = \arcsin m_{y0} = \arccos m_{x0} = \arctan \frac{m_{y0}}{m_{x0}}.$$

If one constructs the graph of m_y/m_x, it can be seen that φ indeed does not depend on the coordinate (participants are not required to do such a verification).



Transient process: m_y increases.



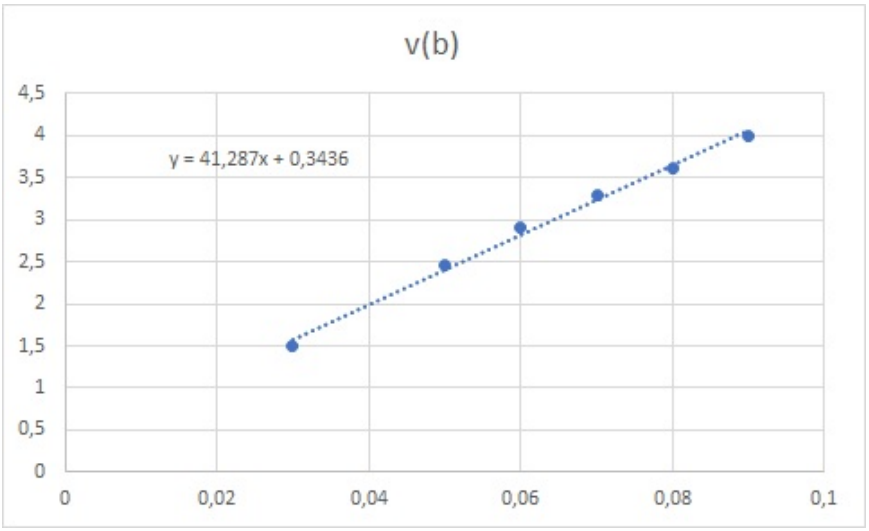
Steady state regime: m_y shape is constant.

If $b > 0.1$, the domain wall behavior changes. Magnetization angle φ increases, magnetic moment rotates around z axis. The domain wall motion is accompanied by its center oscillations. Unlike the case $b < 0.1$, m_y is sometimes greater than m_x (orange and blue graphs swap). Hence $b_{\text{cr}} \approx 0.1$.

Let us measure $v(b)$ and $\varphi(b)$ for $b < 0.1$.

b	m_y	t_1	x_1	t_2	x_2	v	φ
0.03	0.152	40	101,5	50	116,5	1,50	0,152
0.05	0.258	40	134,0	50	158,5	2,45	0,261
0.06	0.316	40	149,0	50	178,0	2,90	0,322
0.07	0.378	40	164,0	50	197,0	3,30	0,387
0.08	0.446	35	159,0	45	195,0	3,60	0,462
0.09	0.525	37	176,0	43	200,0	4,00	0,893

C3. Plot the graph $v(b)$.

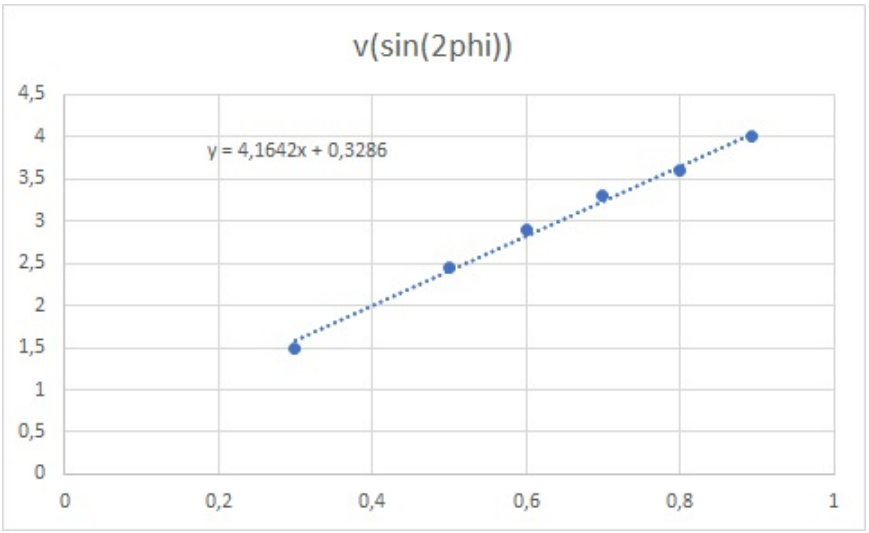


Its slope is

$$\boxed{\frac{\mathrm{d}v}{\mathrm{d}b} = 42}$$

C4. Plot the graph $v(\sin 2\varphi)$. Its slope is $v_c = 4.2$. At $b = b_{\text{cr}}$ the domain wall velocity becomes critical, so that

$$b_{\text{cr}} = \frac{v_c}{\mathrm{d}v/\mathrm{d}b} = 0.10.$$



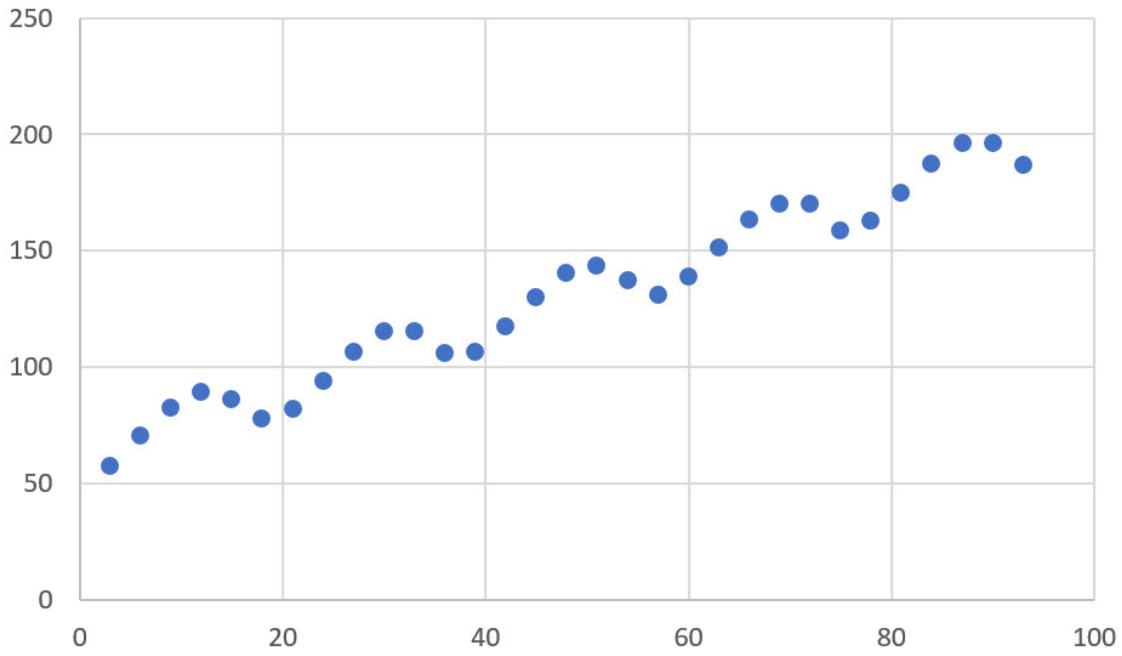
$$\boxed{v_c = 4.2, \qquad b_{\text{cr}} = 0.10}$$

C5. Answer

$$\boxed{v_c = 4.2\, a\omega = 1850\ \text{m/s}; \qquad B_{\text{cr}} = b_{\text{cr}} B_{\text{an}} = 0.083\ \text{T}.}$$

C6. Run the program for $\alpha = 0.2$, $\beta = 100$, $b = 0.2$, $T = 100$, $L = 200$. Plot $m_z(x)$ for $t = 3, 6, 9, 12, \ldots, 90$ and determine x_{center} as an x-coordinate of the intersection of the graph with the x-axis.

t	x_{center}	t	x_{center}	t	x_{center}
3	57.4	33	115.2	63	151.4
6	70.1	36	105.9	66	163.3
9	82.0	39	106.2	69	170.2
12	89.2	42	117.1	72	170.0
15	86.1	45	129.9	75	158.3
18	77.3	48	140.2	78	162.8
21	81.5	51	143.6	81	174.9
24	93.6	54	137.2	84	187.4
27	106.1	57	131.0	87	196.2
30	115.0	60	138.7	90	196.1



The plot $x_{center}(t)$.