Solution

A1. The laser emits pulses with a period of $1/f$. The sample moves with velocity v; therefore, the period of the defects is

$$\boxed{\Lambda = \frac{v}{f}}$$

A2. Using the formula for the wavelength at Bragg resonance, let us express the defect period

$$\Lambda = \frac{\lambda_B}{2n_c}.$$

Substituting into the formula from part A1, we obtain

$$\boxed{v_0 = \Lambda \cdot f = \frac{\lambda_B f}{2n_c} \approx 0.5345 \text{ mm/s}}$$

A3. The minimum of the transmission spectrum $T(\lambda)$ corresponds to the Bragg resonance wavelength.

$$\boxed{\lambda_B = 1543.58 \text{ nm}}$$

A4. Substituting the values into the formula from part A2, we obtain

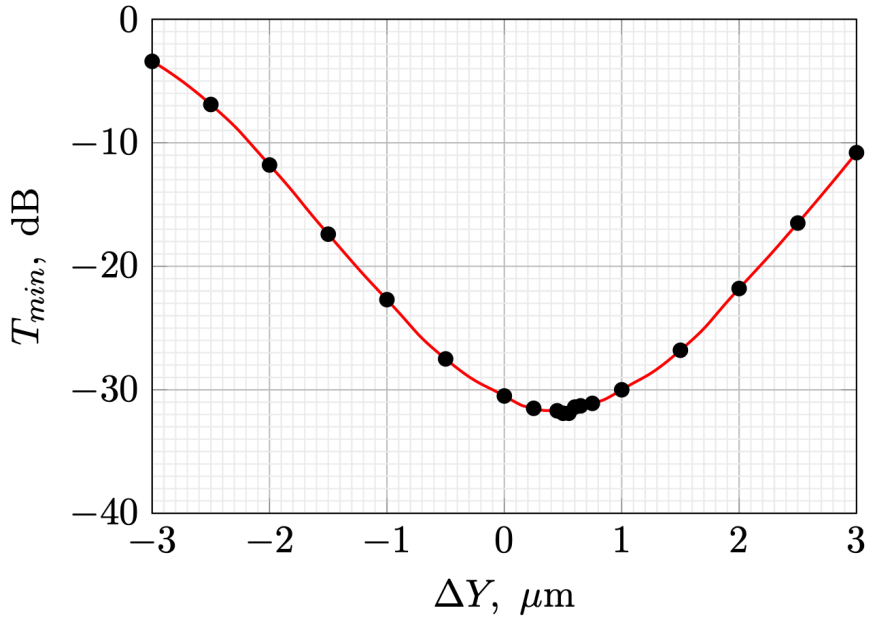
$$\boxed{n_c = \frac{\lambda_B f}{2v_0} = 1.4440}$$

A5. Using the value $\lambda_B = 1550$ nm,

$$\boxed{v_{1550} = \frac{\lambda_B f}{2n_c} = 0.5367 \text{ mm/s}}$$

A6. Let us record the dependence in the specified range with a step of $0.5\,\mu$m. We additionally investigate the vicinity of the extremum of the dependence $T_{\min}(\Delta Y)$.

ΔY, µm	$T_{\min}$, dB	ΔY, µm	$T_{\min}$, dB
−3.0	−3.41	0.55	−31.9
−2.5	−6.90	0.60	−31.4
−2.0	−11.8	0.65	−31.3
−1.5	−17.4	0.75	−31.1
−1.0	−22.7	1.0	−30.0
−0.5	−27.5	1.5	−26.8
0	−30.5	2.0	−21.8
0.25	−31.5	2.5	−16.5
0.45	−31.7	3.0	−10.8
0.5	−31.9		

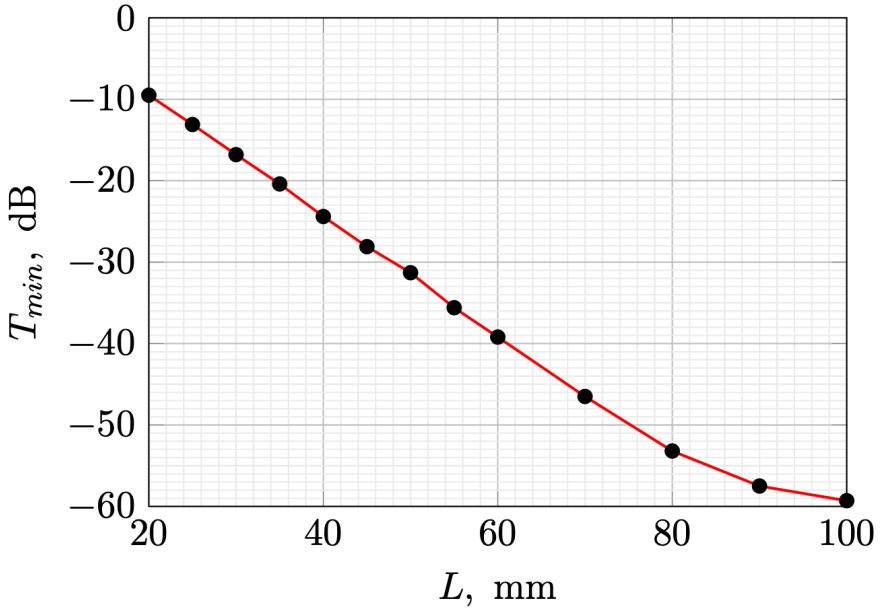


A7. The value ΔY_0 corresponds to the minimum of the dependence measured in the previous part.

$$\boxed{\Delta Y_0 = 0.50\ \mu\text{m}}$$

A8. Let us record the dependence in the range $L \in [20, 100]$ mm. The selected range corresponds to the values of L for which the transmission minimum is clearly pronounced.

L, mm	T_{min}, dB	L, mm	T_{min}, dB
20	−9.5	55	−35.6
25	−13.1	60	−39.2
30	−16.8	70	−46.5
35	−20.4	80	−53.2
40	−24.4	90	−57.5
45	−28.1	100	−59.3
50	−31.3		



The non-linear behaviour of the dependence at large L is due to the background light. From the graph, the slope of the linear section is

$$\boxed{k = -0.742\ \text{dB/mm}}$$

B1. Differentiating the expression for $a'(z)$ with respect to z and substituting $b'(z)$ from the second equation, we obtain

$$a''(z) = i\varkappa b'(z) = \varkappa^2 a(z).$$

$$\boxed{a''(z) - \varkappa^2 a(z) = 0}$$

B2. Let us write the boundary conditions for both components. For the wave propagating in the positive z direction, $a(0) = a_0$. At the right end of the fiber the amplitude of the reflected wave vanishes, so for the wave propagating in the negative z direction $b(L) = 0$.

From part B1 we have a differential equation with particular solutions $a(z) = Ce^{\lambda z}$. Substituting, we obtain the characteristic equation $\lambda^2 - \varkappa^2 = 0$, $\lambda = \pm\varkappa$, so the general solution is

$$a(z) = Ae^{\varkappa z} + Be^{-\varkappa z}.$$

Using the relation from the problem statement,

$$b(z) = \frac{1}{i\varkappa}\, a'(z) = -i\left(Ae^{\varkappa z} - Be^{-\varkappa z}\right).$$

Substituting the boundary conditions, we obtain

$$b(L) = 0 \;\Rightarrow\; B = Ae^{2\varkappa L}, \qquad a(0) = a_0 \;\Rightarrow\; A = \frac{a_0}{1 + e^{2\varkappa L}},$$

$$t' a_0 = a(L) = Ae^{\varkappa L} + Be^{-\varkappa L} = \frac{2a_0 e^{\varkappa L}}{1 + e^{2\varkappa L}} = \frac{a_0}{\cosh(\varkappa L)}.$$

By definition $T = |t'|^2$, therefore

$$\boxed{T = \frac{1}{\cosh^2(\varkappa L)}}$$

B3. *First method*, with approximations for the hyperbolic functions. The slope of the linear part of the graph from A8:

$$T[\text{dB}] = 10\log_{10}\frac{1}{\cosh^2(\varkappa L)}, \qquad \frac{d}{dL}\big(T[\text{dB}]\big) = -\frac{20}{\ln 10}\,\varkappa\tanh(\varkappa L).$$

In terms of the transmission coefficient, $\tanh(\varkappa L) = \sqrt{1 - 1/\cosh^2(\varkappa L)} = \sqrt{1 - T}$. The largest transmission on the linear section gives the lower bound $\tanh(\varkappa L) \approx 0.942$, so we take the hyperbolic tangent to be 1:

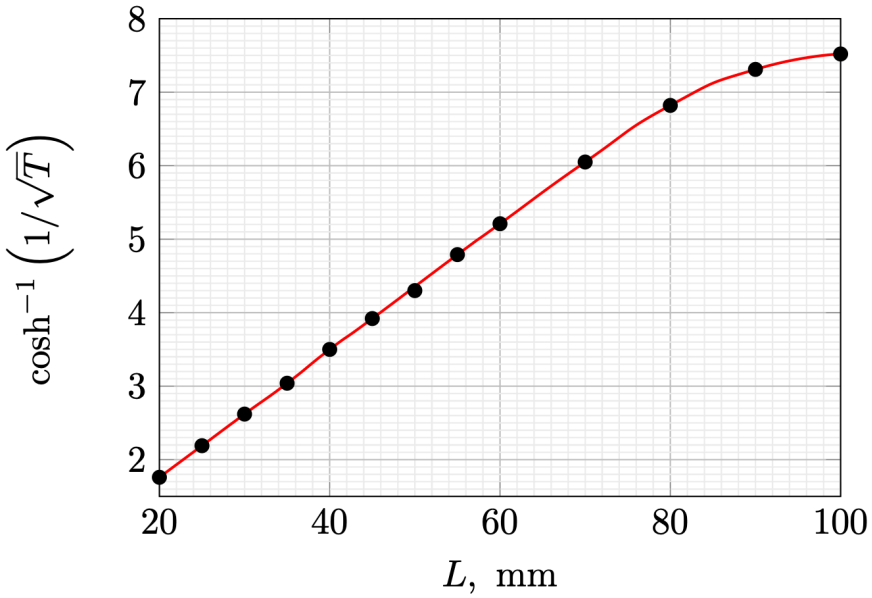
$$\frac{d}{dL}\big(T[\text{dB}]\big) \approx -\frac{20}{\ln 10}\varkappa.$$

The same follows from $\cosh(\varkappa L) \approx e^{\varkappa L}/2$, i.e. $e^{2\varkappa L} \gg 1$: $T[\text{dB}] \approx -\frac{20}{\ln 10}(\varkappa L - \ln 2)$. This is justified on the linear section, since $e^{2\varkappa L} > e^{2\cdot 85.4\cdot 0.02} = 30.45 \gg 1$. Substituting the slope from A8,

$$\boxed{\varkappa_0 = 85.4\ \text{m}^{-1}}$$

Second method, without approximations: we linearise the dependence $T(L)$ from B2, $\cosh^{-1}(1/\sqrt{T}) = \varkappa L$, with T in absolute units, $T = 10^{T[\text{dB}]/10}$.

L, mm	T, dB	$\cosh^{-1}(1/\sqrt{T})$	L, mm	T, dB	$\cosh^{-1}(1/\sqrt{T})$
20	−9.5	1.76	55	−35.6	4.79
25	−13.1	2.19	60	−39.2	5.21
30	−16.8	2.62	70	−46.5	6.05
35	−20.4	3.04	80	−53.2	6.82
40	−24.4	3.50	90	−57.5	7.31
45	−28.1	3.92	100	−59.3	7.52
50	−31.3	4.30			



The slope of the linear part of the graph equals the grating strength:

$$\boxed{\varkappa_0 = 85.9\ \text{m}^{-1}}$$

C1. The total differential of λ_B is $\Delta\lambda_B = 2\Delta n_c \Lambda + 2n_c\Delta\Lambda$. Substituting the temperature dependences of length and refractive index,

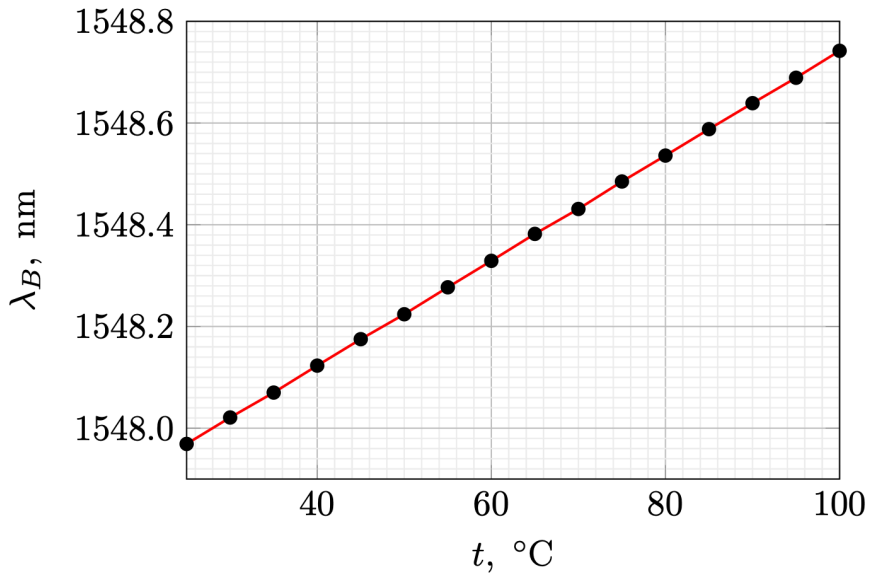
$$\Delta\lambda_B = 2\frac{dn_c}{dt}\Delta t \cdot \Lambda + 2n_c\Lambda\alpha\Delta t = \lambda_B\left(\alpha + \frac{1}{n_c}\frac{dn_c}{dt}\right)\Delta t.$$

$$\boxed{A = \alpha + \frac{1}{n_c}\frac{dn_c}{dt}}$$

C2. Let us record the dependence in the temperature range $t \in [25, 100]$ °C.

t, °C	λ_B, nm	t, °C	λ_B, nm
25	1547.97	65	1548.38
30	1548.02	70	1548.43
35	1548.07	75	1548.49
40	1548.12	80	1548.54
45	1548.18	85	1548.59
50	1548.22	90	1548.64
55	1548.28	95	1548.69
60	1548.33	100	1548.74

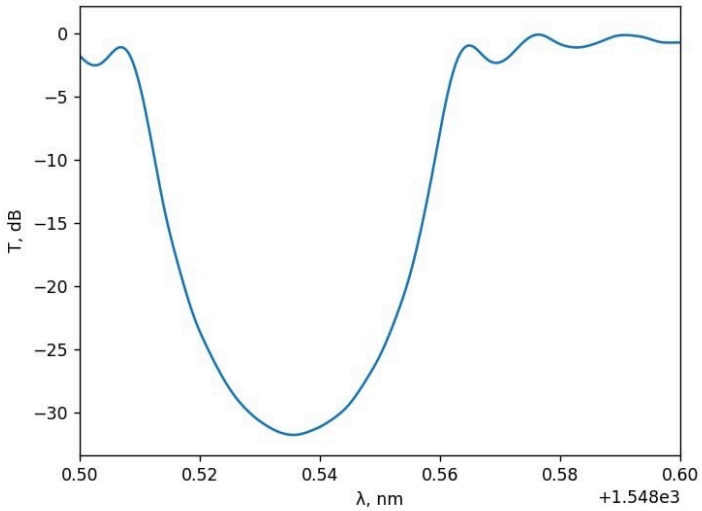
C3.



The slope of the graph is $k = \lambda_B \cdot A = 0.010313$ nm/°C, and the average wavelength is $\lambda_B \approx 1548.4$ nm. Then

$$\boxed{A = 6.66 \cdot 10^{-6}\ ^\circ\mathrm{C}^{-1}}$$

C4. Let us consider the immediate vicinity of the transmission minimum $T(\lambda)$.



The width of the transmission dip is about $\delta\lambda \approx 0.01$ nm; the minimum detectable temperature change is

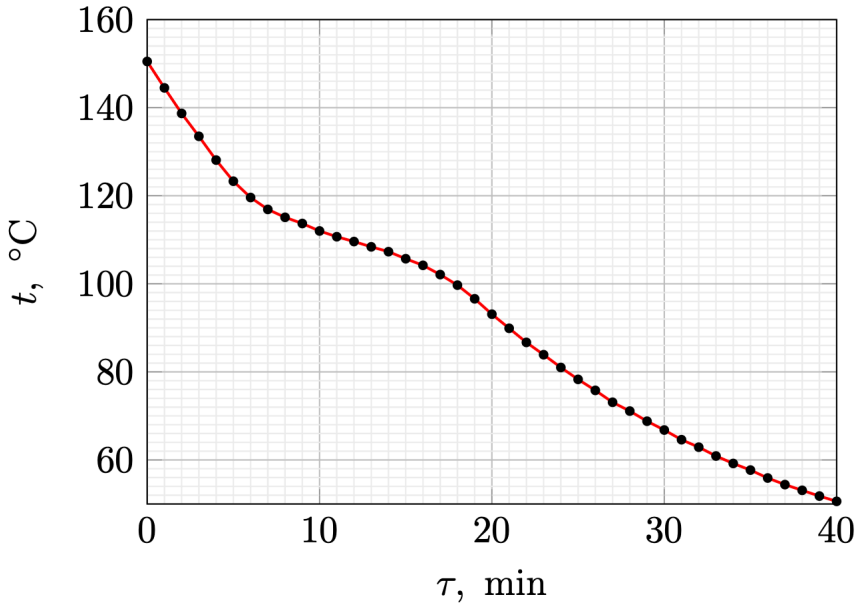
$$\boxed{\Delta t = \frac{\delta\lambda}{\lambda_B A} \approx 1\ ^\circ\mathrm{C}}$$

D1. We record the Bragg wavelength λ_B against time τ and convert it to temperature with the calibration slope from C3:

$$t = \frac{\lambda_B - \lambda_B(0\ ^\circ\mathrm{C})}{\lambda_B A} = \frac{\lambda_B - 1547.71\ \mathrm{nm}}{0.0103\ \mathrm{nm}/^\circ\mathrm{C}}.$$

τ, min	λ_B, nm	t, °C	τ, min	λ_B, nm	t, °C	τ, min	λ_B, nm	t, °C
0	1549.26	150.5	14	1548.81	107.3	28	1548.44	71.1
1	1549.20	144.5	15	1548.80	105.7	29	1548.42	68.8
2	1549.14	138.7	16	1548.78	104.2	30	1548.40	66.8
3	1549.08	133.5	17	1548.76	102.1	31	1548.38	64.6
4	1549.03	128.1	18	1548.74	99.7	32	1548.36	62.9
5	1548.98	123.3	19	1548.70	96.6	33	1548.34	60.9
6	1548.94	119.6	20	1548.67	93.1	34	1548.32	59.2
7	1548.91	116.9	21	1548.64	89.9	35	1548.31	57.7
8	1548.89	115.1	22	1548.60	86.7	36	1548.29	55.9
9	1548.88	113.7	23	1548.57	83.9	37	1548.27	54.4
10	1548.86	112.0	24	1548.54	81.0	38	1548.26	53.1
11	1548.85	110.7	25	1548.52	78.3	39	1548.24	51.8
12	1548.84	109.6	26	1548.49	75.8	40	1548.23	50.6
13	1548.83	108.4	27	1548.46	73.1			

D2.



The graph has a gently sloping section, close to a plateau, corresponding to the change of phase of the LDPE. The transition temperature corresponds to the middle of this section.

$$\boxed{t_0 = 110\ ^\circ\mathrm{C}}$$

E1. The total differential of λ_B is $\Delta\lambda_B = 2\Delta n_c \Lambda + 2n_c \Delta\Lambda$. Substituting the formulas from the problem statement,

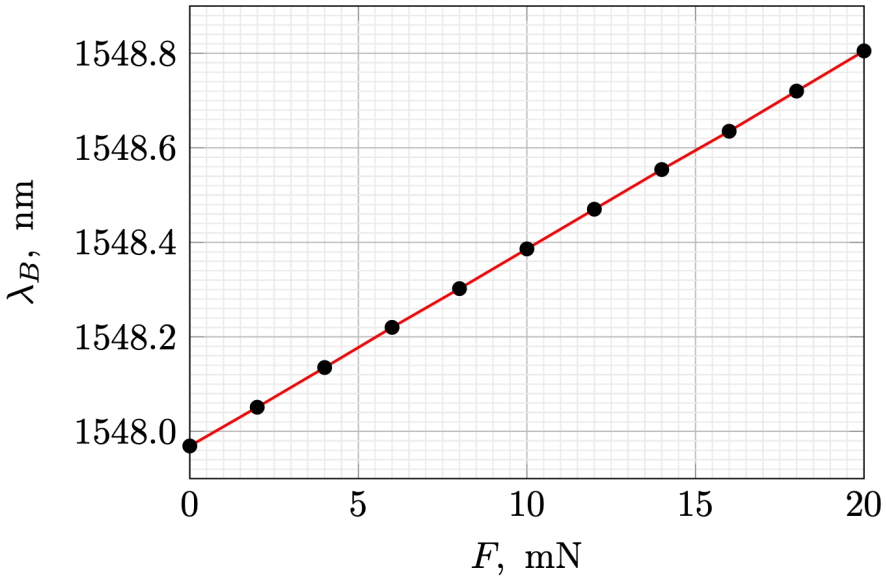
$$\Delta\lambda_B = 2\frac{dn_c}{d\varepsilon}\varepsilon\Lambda + 2n_c\Lambda\varepsilon = \lambda_B\left(1 + \frac{1}{n_c}\frac{dn_c}{d\varepsilon}\right)\varepsilon = \lambda_B\left(1 + \frac{1}{n_c}\frac{dn_c}{d\varepsilon}\right)\frac{F}{ES}.$$

$$\boxed{B = \frac{1}{ES}\left(1 + \frac{1}{n_c}\frac{dn_c}{d\varepsilon}\right)}$$

E2. Let us record the dependence in the range $F \in [0, 20]$ mN.

F, mN	λ_B, nm	F, mN	λ_B, nm
0	1547.97	12	1548.47
2	1548.05	14	1548.55
4	1548.14	16	1548.64
6	1548.22	18	1548.72
8	1548.30	20	1548.81
10	1548.39		

E3.

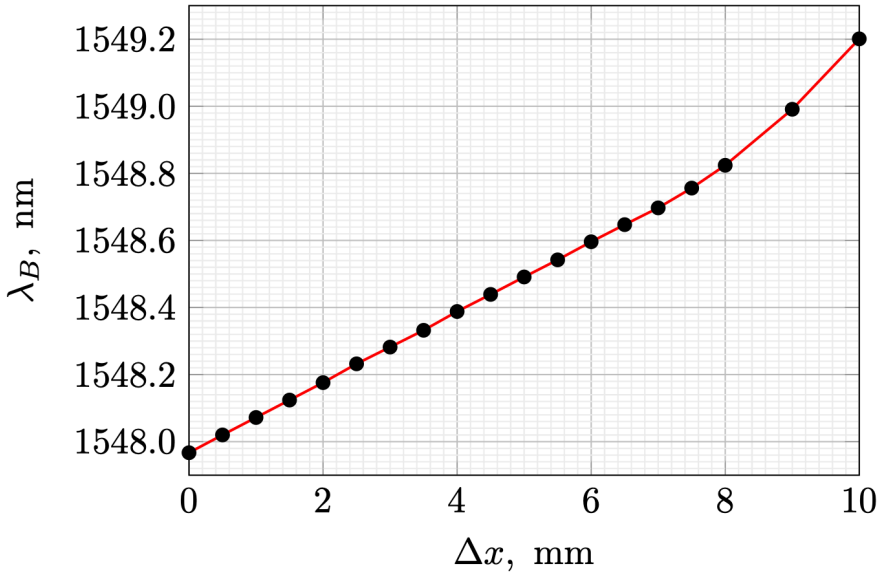


The slope of the graph is $k = \lambda_B \cdot B = 41.80$ nm/N, and the average wavelength is $\lambda_B \approx 1548.4$ nm. Then

$$\boxed{B = 2.70 \cdot 10^{-2}\ \text{N}^{-1}}$$

E4. Let us record the dependence of the Bragg wavelength λ_B on the spring extension Δx.

Δx, mm	λ_B, nm	Δx, mm	λ_B, nm
0	1547.97	5.0	1548.49
0.5	1548.02	5.5	1548.54
1.0	1548.07	6.0	1548.60
1.5	1548.12	6.5	1548.65
2.0	1548.18	7.0	1548.70
2.5	1548.23	7.5	1548.76
3.0	1548.28	8.0	1548.82
3.5	1548.33	9.0	1548.99
4.0	1548.39	10.0	1549.20
4.5	1548.44		



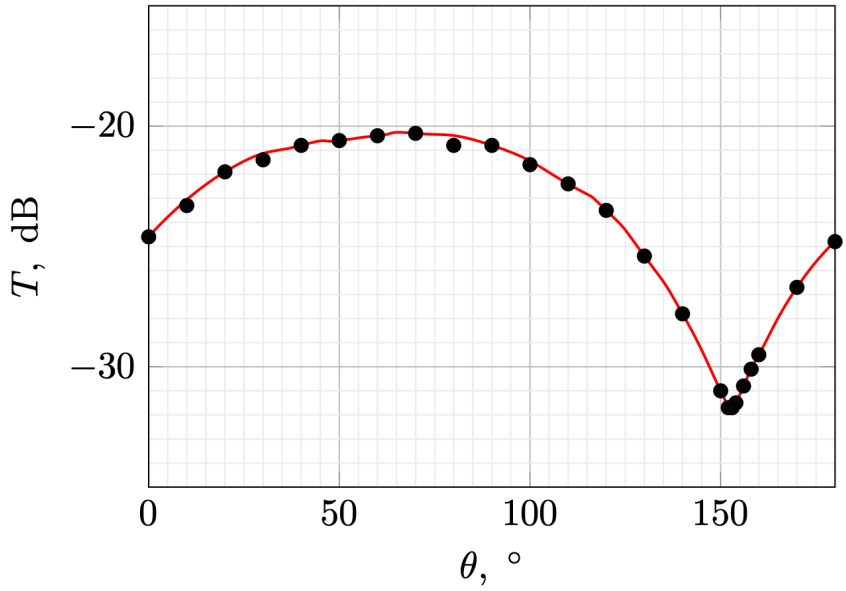
On the linear section of the graph Hooke's law holds for the spring, $F = k\Delta x$. The slope of the linear part is $\kappa = 104.46\ \text{nm/m} = 1.0446 \cdot 10^{-7}$, and

$$k = \frac{\kappa}{\lambda_B B} = 2.50\ \text{N/m}.$$

$$\boxed{k = 2.50\ \text{N/m}}$$

F1. Let us record the depth of the transmission minimum T against the angle θ, looking more closely at the vicinity of the sharp extremum.

θ, °	T, dB	θ, °	T, dB	θ, °	T, dB
0	−24.6	80	−20.8	152	−31.7
10	−23.3	90	−20.8	153	−31.7
20	−21.9	100	−21.6	154	−31.5
30	−21.4	110	−22.4	156	−30.8
40	−20.8	120	−23.5	158	−30.1
50	−20.6	130	−25.4	160	−29.5
60	−20.4	140	−27.8	170	−26.7
70	−20.3	150	−31.0	180	−24.8



The optical field inside the grating can be decomposed into two components, one polarised parallel to the defects and the other perpendicular to them. With E_0 the field amplitude at the input of the SMF and $\theta - \theta_0$ the angle between the polarisation and the defects,

$$E_{0,\parallel} = E_0 \cos(\theta - \theta_0), \qquad E_{0,\perp} = E_0 \sin(\theta - \theta_0).$$

The intensity of each component is proportional to the square of its amplitude, so with transmission coefficients $T_\parallel$ and $T_\perp$ of the components,

$$T = \frac{T_\parallel E_0^2 \cos^2(\theta - \theta_0) + T_\perp E_0^2 \sin^2(\theta - \theta_0)}{E_0^2} = T_\parallel \cos^2(\theta - \theta_0) + T_\perp \sin^2(\theta - \theta_0).$$

According to the problem statement $\varkappa_\parallel < \varkappa_\perp$, therefore $T_\perp < T_\parallel$. From the graph, the minimum of T is reached at $\theta = 153^\circ$; it corresponds to $T_\perp$, the polarisation perpendicular to the defects. Therefore $\theta_0 = \theta - 90^\circ$:

$$\boxed{\theta_0 = \theta - 90^\circ = 63^\circ}$$

F2. The angle $\theta_0 = 63^\circ$ corresponds to polarisation parallel to the defects and $\theta = 153^\circ$ to perpendicular polarisation, with $T_\parallel = -20.5$ dB and $T_\perp = -31.7$ dB. From the formula of B2,

$$\varkappa = \frac{1}{L} \cosh^{-1}\left(\frac{1}{\sqrt{T}}\right).$$

$$\boxed{\varkappa_\parallel = 61.0\ \mathrm{m}^{-1}, \qquad \varkappa_\perp = 86.9\ \mathrm{m}^{-1}}$$

For these angles the minima of the transmission spectra give the Bragg wavelengths:

$$\boxed{\lambda_{B,\parallel} = 1547.95\ \mathrm{nm}, \qquad \lambda_{B,\perp} = 1547.97\ \mathrm{nm}}$$