Solution to Problem 1

At constant falling speed, gravity exactly balances drag, so

$$mg = cv^{\gamma} \qquad \text{(RE2.1)} \qquad \textbf{[1]}$$

Taking common logarithms of both sides and rearranging gives:

$$\lg(v) = \frac{1}{\gamma}\lg(m) + \frac{1}{\gamma}\lg\left(\frac{g}{c}\right) \qquad \text{(RE2.2)}$$

To vary the drag force without changing c , stack cups one inside another. The mass is multiplied by the number of cups, while differences in air resistance depend only on falling speed.

Since only time can be measured, rewrite the equation in terms of fall time from a fixed height H:

$$\lg(T_n) = -\frac{1}{\gamma}\lg(n) - \frac{1}{\gamma}\lg\left(\frac{mg}{c}\right) + \lg H \qquad \text{(RE2.3)} \qquad \textbf{[2]}$$

Here n is the number of nested cups, T_n the fall time of a stack of n cups, and m the mass of each identical cup.

Measurements

Measurements were taken from a height of 1.90 m. The tables give falling times T_1, T_2 and T_3 for one, two and three cups, along with their means and standard deviations for T_1, T_2 and T_3.

Figure 1 shows the graph and its slope, giving $\gamma = 1.84$.

T_1(s)	$<T_1>$(s)	Standard deviation σ (s)	δT (s)
2.51	2.51	0.089	0.02
2.50			
2.53			
2.41			
2.50			
2.47			
2.59			
2.40			
2.46			
2.47			
2.40			
2.46			
2.50			
2.44			
2.60			
2.63			
2.57			
2.62			
2.47			
2.75			

T_2(s)	$<T_2>$(s)	Standard deviation σ (s)	δT (s)
1.72	1.70	0.058	0.012
1.71			
1.63			
1.75			
1.72			
1.62			
1.75			
1.71			
1.75			
1.75			
1.72			
1.69			
1.72			
1.66			
1.59			
1.59			
1.62			
1.59			
1.66			
1.66			

T_3(s)	$<T_3>$(s)	Standard deviation σ (s)	δT (s)
1.31	1.38	0.046	0.010
1.38			
1.41			
1.41			
1.31			
1.38			
1.38			
1.38			
1.37			
1.34			
1.34			
1.41			
1.32			
1.44			
1.40			
1.48			
1.44			
1.47			
1.44			
1.37			

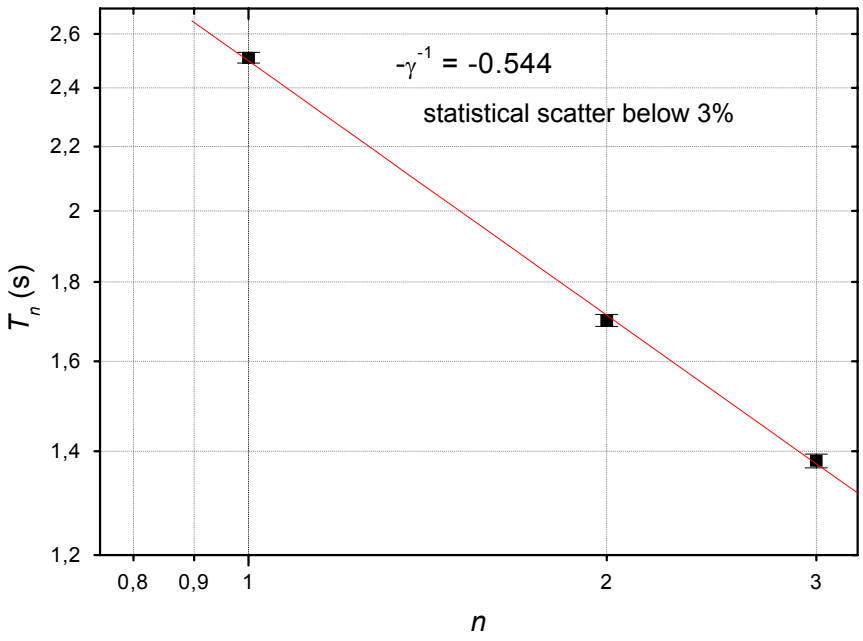


Figure RE2.1: Log-log plot showing the power-law dependence of falling time on cup count. Error bars give the total timing error.

Error analysis

In the tables, the standard deviation was calculated as:

$$\sigma = \sqrt{\frac{\sum_i \left(m_i - \langle m \rangle\right)^2}{9}} \qquad \text{where} \qquad \langle m \rangle = \frac{1}{n}\sum_i x_i \qquad \text{(RE2.4)}$$

while the random error was calculated as

$$\delta m = \frac{\sigma}{\sqrt{n}} \qquad \text{(RE2.5)}$$

The total measurement error is

$$E = \sqrt{e_T{}^2 + \delta T^2} \qquad \text{(RE2.6)}$$

Here $e_T = 0.01\ s$. Thus the total errors for each mass are:

$E_{T1} = 0.02\ s$
$E_{T2} = 0.015\ s$
$E_{T3} = 0.014\ s$

Find the upper error bound for the exponent γ:

$$\gamma = \frac{\log \frac{m_3}{m_1}}{\log\left(\frac{T_1}{T_3}\right)} \qquad \text{(RE2.7)}$$

As mass error is negligible, retain only the second term. The increment is:

$$\Delta\gamma = \frac{1}{2}\ln\frac{m_3}{m_1}\left\{\frac{1}{\ln\frac{T_1-\Delta T_1}{T_3+\Delta T_3}} - \frac{1}{\ln\frac{T_1+\Delta T_1}{T_3-\Delta T_3}}\right\} \qquad \text{(RE2.8)}$$

Evaluation gives the upper error bound for γ:

$$\frac{\Delta\gamma}{\gamma} = \frac{1}{2}\frac{\left\{\frac{1}{\ln\frac{T_3-\Delta T_3}{T_1+\Delta T_1}} - \frac{1}{\ln\frac{T_3+\Delta T}{T_1-\Delta T_1}}\right\}}{\ln\frac{T_3}{T_1}} \qquad \text{(RE2.9)}$$

Substituting the graph extreme values, T_1 and T_3, gives:

$$\frac{\Delta\gamma}{\gamma} = 0.086$$

Thus the measured exponent is $\gamma = 1.84 \pm 0.16$ (± 9 %)