EXPERIMENTAL EXAMINATION A (10 points)

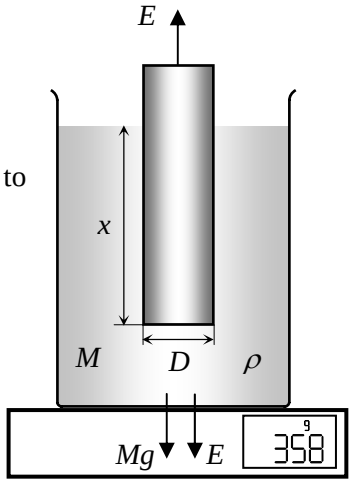
Archimedes' principle. Measuring the density of a liquid.

Equipment.

- Plastic cup.
- Unknown liquid (apple juice).
- Plastic cylinder with suspension hook.
- Base with vertical rod, threaded rod and sleeve, bosshead clamp and thread for suspending the cylinder.
- Digital balance.
- Adhesive tape.
- Marker pen and ruler.

Theoretical basis.

Archimedes' principle states that *a body immersed in a fluid experiences an upward buoyant force equal to the weight of the displaced fluid.*



In the experiment shown, a vertical cylinder of diameter D is immersed to length x, displacing volume

$$V = \pi (D/2)^2 x .$$

The upward buoyant force is therefore

$$E = \rho V g = \rho \pi (D/2)^2 g\, x ,$$

where ρ is the liquid density and g is gravitational acceleration.

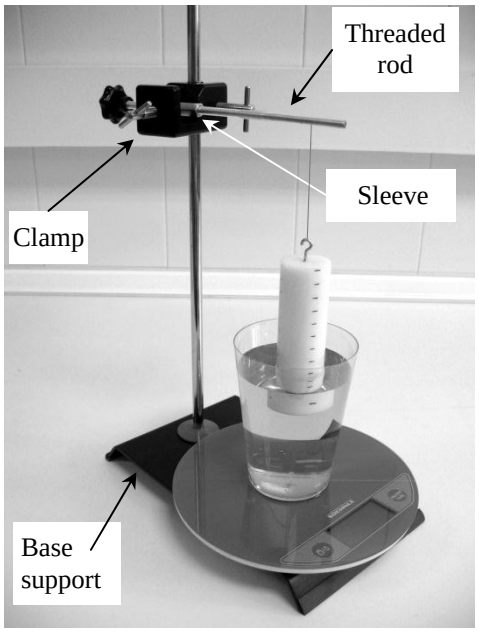
This force can be measured by placing the system on a balance, which reads apparent mass: for total supporting force F, it reads $M_{ap} = F/g$. By action and reaction, the force on the balance is the cup-and-liquid weight Mg plus the force E exerted by the cylinder on the liquid. Hence its apparent-mass reading is

$$M_{ap} = M + \frac{E}{g} = M + \rho \frac{\pi D^2}{4} x .$$

Data: Cylinder diameter: $D = (36{,}0 \pm 0{,}1)$ mm**.**

Experimental procedure.

For precise measurements of immersed length x, use the ruler and marker to draw one or more scales on the cylinder's side.



Place the balance on the stand base and the cup on the balance. Suspend the cylinder over the cup's centre using thread wound around the threaded rod. Tape the thread's end to the rod so it cannot slip.

Attach the assembly to the vertical rod using the hexagonal threaded sleeve and bosshead clamp (see the photograph).

Adjust immersed length x by turning the threaded rod to wind or unwind the thread.

Choose the initial liquid level so the cylinder can be immersed to at least x = 10 cm without touching the bottom or overflowing the cup.

Before measuring, remove the cup and press “ON”. Check that the balance reads grams (“g” at the top of the display). Otherwise press the left button until it does. Then centre the cup on the balance.

Precautions and advice:

- The balance has four feet underneath. All four must rest on the same horizontal surface—the stand base—for correct readings.
- Do not let the cylinder rub against the cup walls or touch the bottom.
- An air bubble beneath or beside the cylinder can distort the measurements.
- The balance has a “TARE” function: pressing the right button while it is on sets the present load as the zero reference. The *tare* must stay unchanged during measurements, so do not press “TARE” while measuring.
- The balance switches off automatically after about four minutes without a change in reading. If this happens, switch it on again under the same initial conditions to preserve the tare. To avoid this, complete each measurement series without long delays.
- To switch off, hold the right “OFF” button for a few seconds.
- Leave the workstation clean and tidy, as you found it.
- **Warning:** Take great care not to overturn the cup and spill the liquid. **This could ruin your work.**

Questions.

a) For a series of x values, record the apparent-mass reading M_{ap}. Present the data in a table. (2 points)

b) Plot the measured points $\left(x, M_{ap}\right)$ on graph paper. (2 points)

c) Fit a straight line to the points. (2 points)

d) From the fit, find the liquid density ρ . (1 point)

e) Considering only the stated cylinder-diameter uncertainty ΔD = 0,1 mm, calculate the liquid-density uncertainty $\Delta\rho_D$. (3 points)

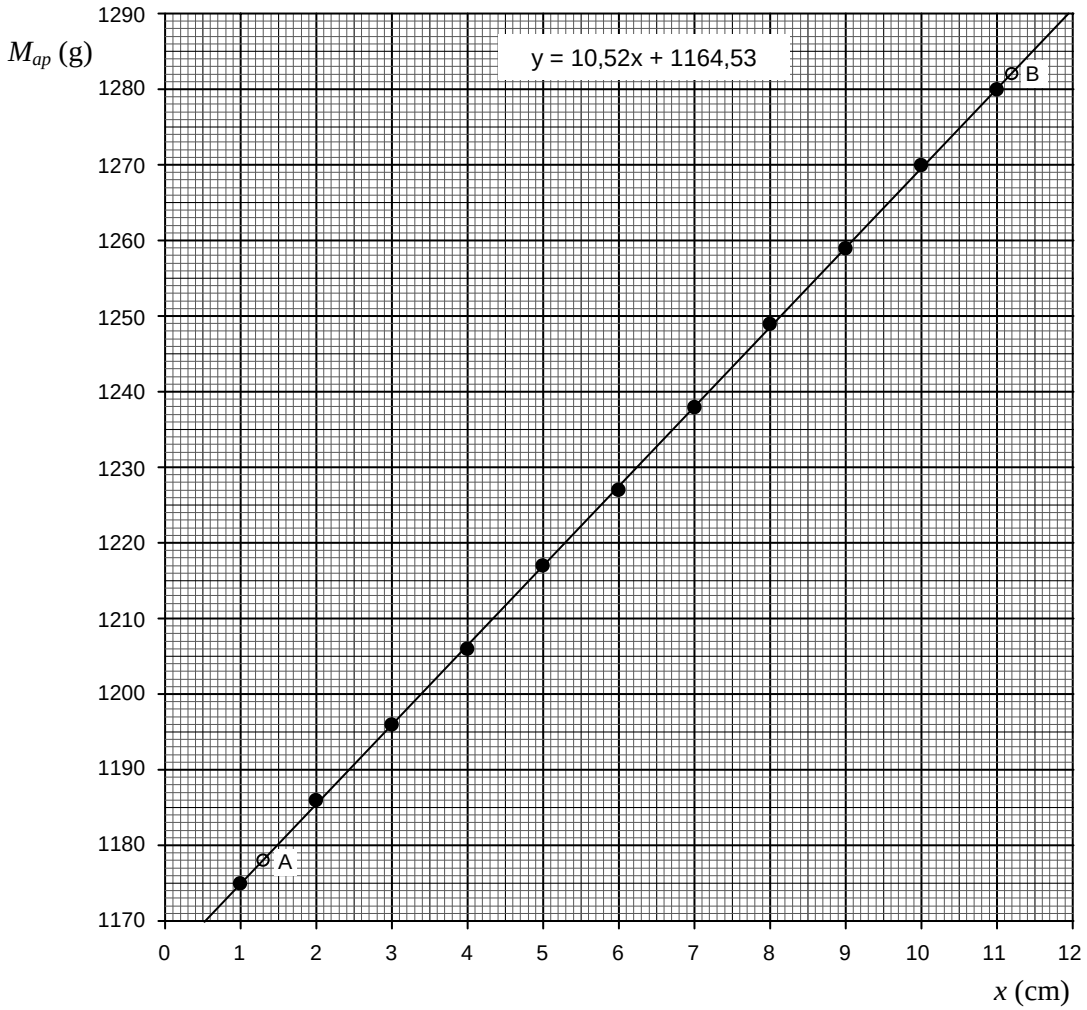
EXPERIMENTAL TEST A

Archimedes' principle. Measuring the density of a liquid. **Solution**

a) Measurement table:

x **(cm)**	1	2	3	4	5	6	7	8	9	10	11
M_{ap} **(g)**	1175	1186	1196	1206	1217	1228	1238	1249	1259	1270	1280

b) The graph below resembles one drawn on graph paper. The $x = 0$ point is excluded because visual alignment can give it an appreciable systematic offset relative to the others.



c) As expected, the data lie very close to a straight line. Its slope can be found manually by drawing the best-fitting line and reading two widely separated points, such as A and B marked at grid intersections. Their coordinates are:

$$(x_A;\, y_A) = (1{,}30\,\text{cm};\, 1178{,}0\,\text{g})$$

$$(x_B;\, y_B) = (11{,}20\,\text{cm};\, 1282{,}0\,\text{g})$$

The slope is therefore

$$p = \frac{y_B - y_A}{x_B - x_A} = \frac{104{,}0\,\text{g}}{9{,}90\,\text{cm}} \qquad \boxed{p = 10{,}51\ \text{g/cm} = 1{,}051\ \text{kg/m}}$$

A least-squares fit gives a very similar result:

$$\boxed{p = 10{,}52\ \text{g/cm} = 1{,}052\ \text{kg/m}}$$

d) From the model in the question, the slope of M_{ap} versus x is

$$p = \rho\frac{\pi D^2}{4} \quad \Rightarrow \quad \rho = \frac{4p}{\pi D^2}$$

Using the latter slope gives

$$\boxed{\rho = 1{,}034\,\text{g/cm}^3 = 1{,}034\times10^3\,\text{kg/m}^3}$$

e) The stated cylinder diameter is $D = (36{,}0 \pm 0{,}1)$ mm, with uncertainty $\Delta D = 0{,}1$ mm. The propagated density uncertainty $\Delta\rho_D$ can be calculated numerically from the extreme diameters

$$D_{\min} = 3{,}59\ \text{cm} \qquad D_{\max} = 3{,}61\ \text{cm}$$

$$\Delta\rho_D = \frac{1}{2}(\rho_{\max} - \rho_{\min}) = \frac{2p}{\pi}\left(\frac{1}{D_{\min}^2} - \frac{1}{D_{\max}^2}\right) \qquad \boxed{\Delta\rho_D = 0{,}006\ \text{g/cm}^3}$$

The same result follows more elegantly, though no more precisely, by taking absolute increments in the density expression:

$$\rho = \frac{4p}{\pi D^2} \quad \Rightarrow \quad \Delta\rho_D = 2\frac{4p}{\pi D^3}\Delta D = 2\rho\frac{\Delta D}{D} = 0{,}006\ \text{g/cm}^3$$

Note: A full error analysis would also include slope uncertainty Δp and its propagated density error $\Delta\rho_p$. However, the data are so well aligned that manual estimates from the steepest and shallowest reasonable lines are difficult. Error bars can be estimated from measurement resolution, $\Delta x \approx 1$ mm and $\Delta M_{ap} \approx 1$ g ; these are as small as the plotted point radii. Alternatively allow for drawing the best line and reading A and B by hand. Either approach gives a vertical uncertainty of about one small grid square:

$$\Delta(y_B - y_A) = 1\ \text{g} \quad \Rightarrow \quad \Delta p = 0{,}09\ \text{g/cm}$$

The analytically calculated slope standard error is $\varepsilon_p = 0{,}03$ g/cm, so the preceding Δp corresponds to a very high confidence level, above 95%.

Density is directly proportional to slope, so relative uncertainty propagates as:

$$\Delta\rho_p = \rho\frac{\Delta p}{p} = 0{,}009\ \text{g/cm}^3$$

In total, $\Delta\rho = \left(\Delta\rho_D^2 + \Delta\rho_p^2\right)^{1/2} \qquad \Delta\rho = 0{,}011\,\text{g/cm}^3$

EXPERIMENTAL EXAMINATION B (10 points)

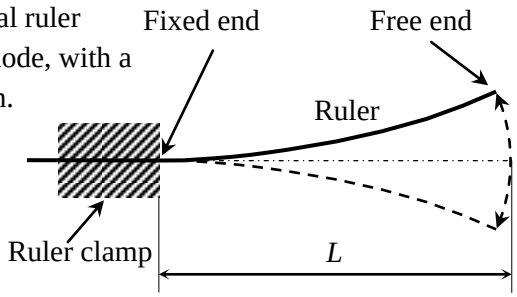
Transverse vibration of a metal ruler.

Equipment.

- Metal ruler.
- Aluminium ruler clamp.
- G-clamp to fix the assembly to the table.
- Stroboscope with external power supply.
- Multimeter with frequency function.
- Black card.

Theoretical basis.

This experiment studies transverse oscillations of a metal ruler versus its free length L. In particular, study the fundamental mode, with a node at the fixed end and an antinode at the free end, as shown.



The ruler's vibration frequency f is expected to depend on length as

$$f = K L^n , \qquad (1)$$

where n is an integer or half-integer, and K depends on the ruler's material and transverse dimensions.

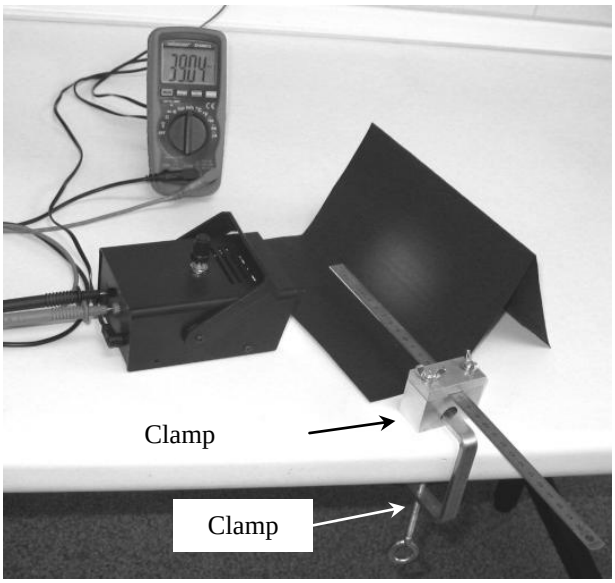
The aim is to determine n and K by measuring f versus L.

Experimental procedure.

The aluminium clamp holds the ruler with adjustable free length L. Loosen the two top wing nuts, slide the ruler to the required length and retighten them.

Fix the aluminium clamp to the table edge with the G-clamp, as in the photograph.

The ruler oscillates too fast for manual timing. Use *stroboscopic* illumination: a source emits very brief periodic flashes at adjustable frequency F. When $F = f$, successive flashes illuminate the ruler at the same phase, making it appear stationary.



The ruler also looks stationary if F is a submultiple of f, that is, $F = f/2,\ f/3,\ f/4...$ since it completes 2, 3, 4… oscillations between two successive flashes. Conversely, if F is a multiple of f, namely, $F = 2f, 3f...$ it appears stationary at 2, 3… positions of its oscillation.

Adjust stroboscope frequency F with the ten-turn potentiometer on top; measure it with the multimeter set to "Hz". Connect the probes to the red and black terminals on the back.

Connect the power supply to the mains and its 12 V output to the back of the stroboscope.

The ruler's edge is painted diffuse white for contrast. View it against a dark background. Fold the black card and place it below and behind the ruler, opposite the stroboscope, as shown.

Data: Up to 99,99 Hz, frequency-meter accuracy is ±1,5% of the reading.

Comments and advice:

- The ruler's real oscillation is damped: amplitude gradually falls until it stops at equilibrium. Even with F equal to f, it will slowly approach equilibrium rather than appear perfectly still.
- Place the ruler correctly in the groove between the two aluminium clamp pieces. Keep the top piece parallel to the bottom so the full ruler width is clamped uniformly.
- The G-clamp must hold the aluminium piece firmly so it cannot vibrate.
- When not measuring, turn off the stroboscope supply to prevent overheating and disturbing other participants.
- The multimeter switches off after a few minutes. Press the yellow "Mode" button to switch it on again.

Questions.

a) With free length L = 14,0 cm, find and record all flash frequencies F_1, F_2, F_3… for which the ruler appears stationary in one or two positions. Search the full range, approximately 6–50 Hz. Which is the actual vibration frequency f? (2 points)

b) Measure vibration frequency f versus length L from L_{min} = 10,0 cm to L_{max} = 22,0 cm. (2 points)

c) Using a suitable graph and fit, deduce exponent n in equation (1). (2 points)

d) Determine K using a suitable graph and fit, remembering that n must be an integer or half-integer. (2 points)

e) Estimate the uncertainty in K. (2 points)

EXPERIMENTAL TEST B

Transverse vibration of a metal ruler. Solution

a) With L = 14,0 cm, sweeping up from low flash frequencies gives a stationary ruler at

$$F_1 = 7{,}17 \text{ Hz}, \quad F_2 = 10{,}72 \text{ Hz}, \quad F_3 = 21{,}45 \text{ Hz}.$$

It appears double at

$$F_4 = 42{,}98 \text{ Hz}.$$

As expected, these four values are approximately in the ratio 1/3 : 1/2 : 1 : 2. The vibration frequency at this length is therefore $f = F_3 = 21{,}45$ Hz.

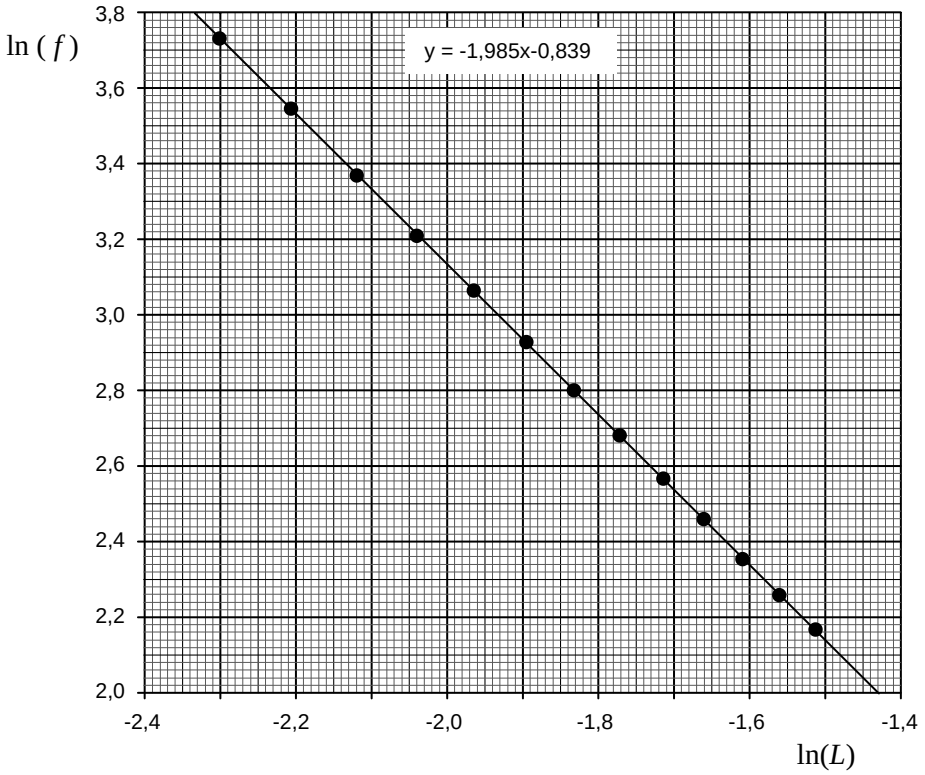
b) The following table lists systematic measurements of f versus L and other values needed later.

L (m)	0,100	0,110	0,120	0,130	0,140	0,150	0,160	0,170	0,180	0,190	0,200	0,210	0,220
f (Hz)	41,76	34,70	29,03	24,75	21,42	18,71	16,47	14,59	13,04	11,70	10,53	9,56	8,74
ln(L)	-2,303	-2,207	-2,120	-2,040	-1,966	-1,897	-1,833	-1,772	-1,715	-1,661	-1,609	-1,561	-1,514
ln(f)	3,732	3,547	3,368	3,209	3,064	2,929	2,802	2,680	2,568	2,460	2,354	2,258	2,168
$1/L^2$ (m^{-2})	100,00	82,64	69,44	59,17	51,02	44,44	39,06	34,60	30,86	27,70	25,00	22,68	20,66

c) Taking logarithms in any base linearises equation (1):

$$\ln(f) = \ln(K) + n\ln(L) \qquad (2)$$

The required n is therefore the slope of ln(f) versus ln(L). The following graph resembles one drawn on graph paper at a reduced scale.



As expected, the points are well aligned. Least squares gives slope $p = -1{,}985$, so the required integer or half-integer exponent is clearly

$$\boxed{n = -2}$$

Thus the relation between f and L is

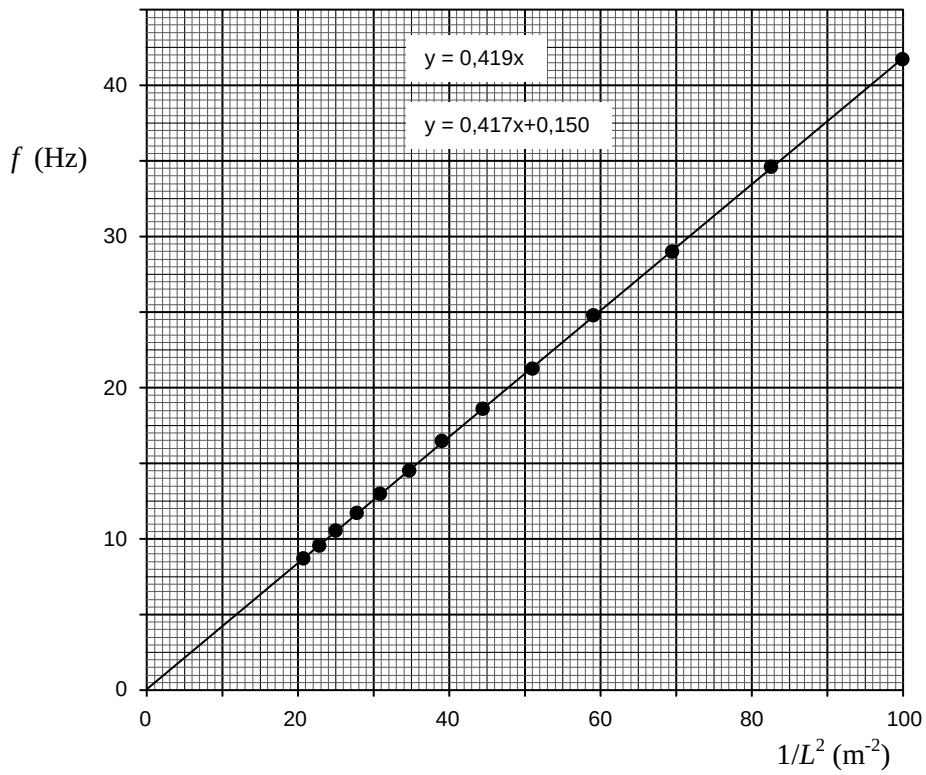
$$f = \frac{K}{L^2} \qquad (3)$$

d) An initial approximation to K follows from the preceding fit, since by (2) its intercept is

$$\ln(K) = -0{,}839 \qquad \Rightarrow \qquad K = 0{,}432 \text{ m}^2 \text{ s}^{-1}$$

This fit uses n close to -2 but not exactly equal to it, introducing a small deviation in K.

It is more accurate to return to the theoretical model with $n = -2$, equation (3): K is the slope of f versus $1/L^2$. The graph below confirms a very good straight line through the origin with slope $p = 0{,}419$ m^2 s^{-1}. Allowing a free intercept gives almost the same slope to three significant figures: $p = 0{,}417$ m^2 s^{-1}.



The first fit follows the proposed model and is considered more reliable, giving

$$\boxed{K = 0{,}419 \text{ m}^2 \text{ s}^{-1}}$$

e) The uncertainty ΔK equals the preceding slope uncertainty. A manual estimate uses the steepest and shallowest lines reasonably fitting the data. Scatter is almost absent (slope standard error $\varepsilon_p = 0{,}0006\,\mathrm{m^2\,s^{-1}}$, only 0,14%), so estimate the uncertainty of each point, its error bars.

On the horizontal axis, the absolute and relative uncertainties in $1/L^2$ are

$$\Delta\left(\frac{1}{L^2}\right) = 2\frac{\Delta L}{L^3} \qquad \Rightarrow \qquad \frac{\Delta\left(1/L^2\right)}{1/L^2} = 2\frac{\Delta L}{L}$$

A slightly pessimistic uncertainty for length L, $\Delta L = 0{,}5\,\mathrm{mm}$ gives $100\Delta L/L$ from 0,5% at $L = 10$ cm to 0,23% at $L = 22$ cm. The relative uncertainty in $1/L^2$ is twice this: from 1% at the highest $1/L^2$ to 0,5% at the lowest. These errors should be random, affecting the slope by less than 1%.

The uncertainty in frequency f on the vertical axis is directly set by the meter accuracy, 1,5%. A possible calibration offset is systematic.

A reasonable total uncertainty of the slope, hence K, is about 2%, giving

$$\boxed{\Delta K = 0{,}008\ \mathrm{m^2\,s^{-1}}}$$

Note on the constant K:

It can be shown that

$$K = 0{,}1615\sqrt{\frac{Y}{\rho}}\,b$$

where Y and ρ are the material's Young modulus and density, and b is the ruler thickness.

Here the ruler is steel, with

$$Y \approx 2{,}1\times10^{11}\,\mathrm{Pa}$$

$$\rho \approx 7{,}8\times10^{3}\,\mathrm{kg/m^3}$$

$$b = 0{,}50\ \mathrm{mm}$$

This gives

$$K \approx 0{,}42\ \mathrm{m^2\,s^{-1}}$$

Our experimental result agrees very well with this value.