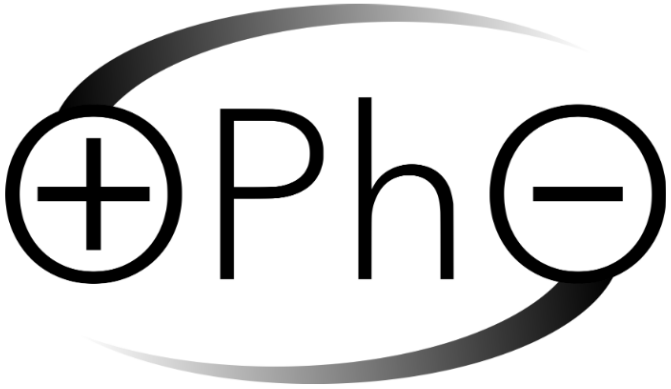
2021 Online Physics Olympiad: Invitational Contest



Experimental Exam

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Solution

Part One

Problem 1.1

We have:

$$A(\theta) \propto \left(\frac{\sin\beta}{\beta}\right)\cos\left(\frac{\pi d}{\lambda}\theta\right) \tag{4}$$

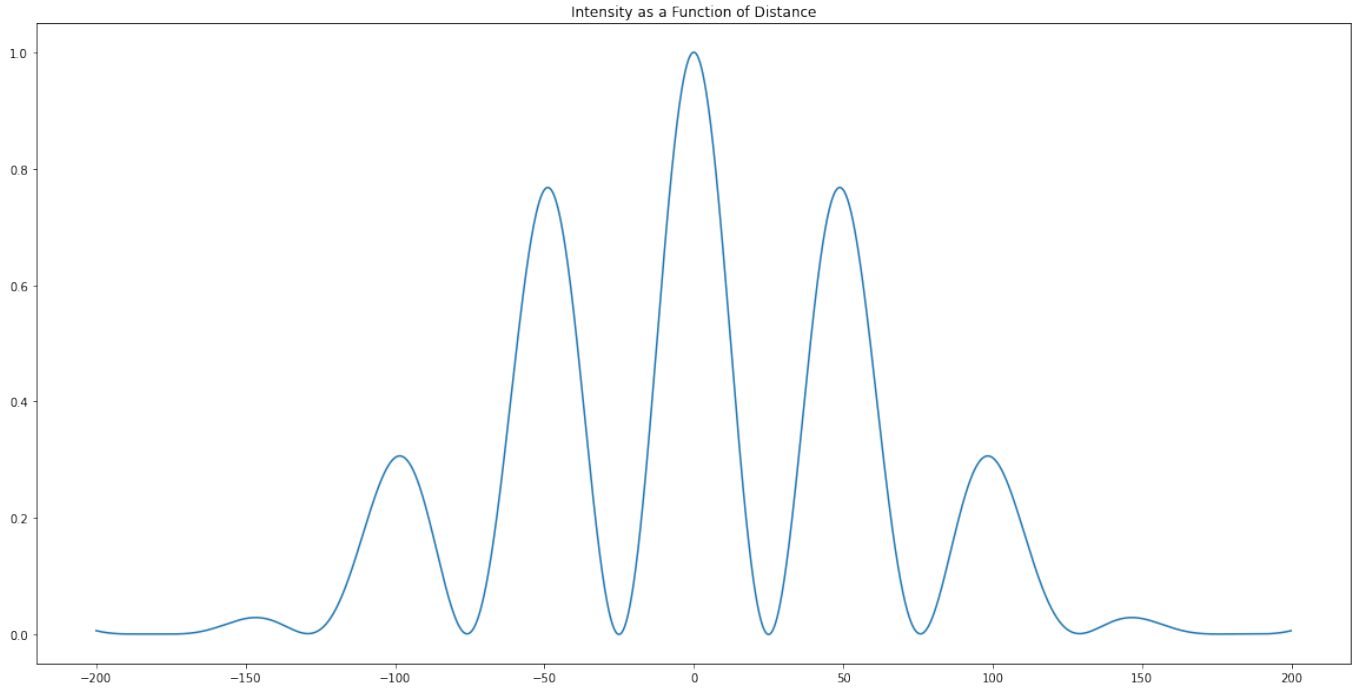
where $\beta = \dfrac{\pi w \sin\theta}{\lambda}$.

Problem 1.2

We will let $w = 3\lambda$ and $d = 10\lambda$, and let the distance between the slits and the wall be $D = 500\lambda$. This corresponds with setting the slits to be

```
slits = [
      [-5*wavelength, 3*wavelength, wavelength, wavelength, 1],
      [5*wavelength, 3*wavelength, wavelength, wavelength, 1]
    ]
```

and gives the following pattern:



The first minimum is at $(25.0 \pm 0.1)\lambda$. Note that the uncertainty comes from the fact that the for loop iterates through locations on the screen in intervals of 0.1λ.

Theoretically, the first minimum occurs when $\cos\left(\dfrac{\pi d}{\lambda}\theta\right) = 0$, which implies $\dfrac{\pi d\theta}{\lambda} = \dfrac{\pi}{2}$, which occurs when $\theta = \dfrac{\lambda}{2d}$. Using the approximation $\theta \approx \dfrac{x}{D}$, we get the first minimum to be:

$$x_{\min} = \frac{D}{2d}\lambda = 25\lambda. \tag{5}$$

However, the second minimum occurs at $(75.9 \pm 0.1)\lambda$, which disagrees with the theoretical $x_{\min,2} = 3x_{\min} = 75\lambda$. Even after accounting for non-small angles, i.e. $x_{3,\min} = D\arctan\left(\dfrac{3\lambda}{2d}\right) = 74.4\lambda$, it still doesn't agree. This is because the code doesn't adjust for thick slits properly.

Problem 1.3

For an angle θ, phasors of the two patterns will each have an amplitude $A_0(\theta)$, but are an angle $\frac{2\pi d}{\lambda}\theta$ apart. Therefore:

$$A = A_0(\theta)\cos\left(\frac{\pi d}{\lambda}\theta\right) \tag{6}$$

Note that we have already verified this experimentally using the code in the previous question. We just need to show that the interference pattern of a single (wide) slit is given by:

$$A(\theta) \propto \frac{\sin\beta}{\beta}, \tag{7}$$

which we can do similarly to what we did above by looking at the minimum and/or maximum points.

Problem 1.4

The separations are $d, 2d, \ldots, 2^{k-1}d$. Therefore, using the previous result recursively, we get:

$$A = A_0(\theta)\cos\left(\frac{\pi d}{\lambda}\theta\right)\cos\left(\frac{2\pi d}{\lambda}\theta\right)\cdots\cos\left(\frac{2^{k-1}\pi d}{\lambda}\theta\right) \tag{8}$$

Let us consider the case where $d = w$ and take the limit $k \to \infty$ while keeping $2^k w$ constant. On the left hand side, we have a single slit of width $2^k d$, and we know the pattern is

$$A = \frac{\sin x}{x} \tag{9}$$

if we let $x = \frac{\pi(2^k d)\theta}{\lambda}$. On the right hand side, we have the same scenario as the previous part:

$$A = A_0(\theta)\cos(x/2)\cos(x/4)\cdots\cos\left(x/2^k\right).$$

As we take $k \to \infty$, we have $d \to 0$, implying A_0 goes to 1. Therefore, we get the desired relationship.

Part Two

Problem 2.1

Any aperture pattern can be represented by an aperture function, where a slit centered at x_0 that reduces the amplitude to f of the original can be represented by $f \cdot \delta(x - x_0)$ where $\delta(x)$ is the delta function.

It turns out that taking the Fourier transform of the aperture function can give the amplitude function. It may seem strange that we are transforming a variable with dimension of length x_1 (Which represents the location of the slits) to another dimension of length x_2 (which represents the location at the screen), when Fourier transforms typically take variables to their inverses (i.e. time to frequency). However, this is not the case. We are actually taking the transform from x_1 to $\frac{x_2}{\lambda D}$.

While we will not go into the details, it turns out taking the Fourier transform of the amplitude function will also get us the aperture function (i.e. transforming x_2 into x_1).

For a double slit, the intensity function is $\cos^2\left(\pi d\frac{x}{\lambda D}\right) = \cos^2(\pi d x_2)$. We can use the identity $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ to write:

$$\cos^2(\pi d \cdot x_2) = \frac{1}{2} + \frac{1}{2}\cos(2\pi d \cdot x_2). \tag{10}$$

Taking the Fourier Transform, we get:

$$\hat{f}\left[\frac{1}{2}+\frac{1}{2}\cos(2\pi d\cdot x_2)\right]=\hat{f}\left[\frac{1}{2}\right]+\hat{f}\left[\frac{1}{2}\cos(2\pi d\cdot x_2)\right] \tag{11}$$

$$=\frac{1}{2}\hat{f}[1]+\frac{1}{2}\hat{f}[\cos(2\pi d\cdot x_2)] \tag{12}$$

$$=\frac{1}{2}\delta(x_1)+\frac{1}{2}\left(\frac{\delta\left(x_1-\frac{2\pi d}{2\pi}\right)+\delta\left(x_1+\frac{2\pi d}{2\pi}\right)}{2}\right) \tag{13}$$

$$=\frac{1}{2}\delta(x_1)+\frac{1}{4}\delta\left(x_1-d\right)+\frac{1}{4}\delta\left(x_1+d\right) \tag{14}$$

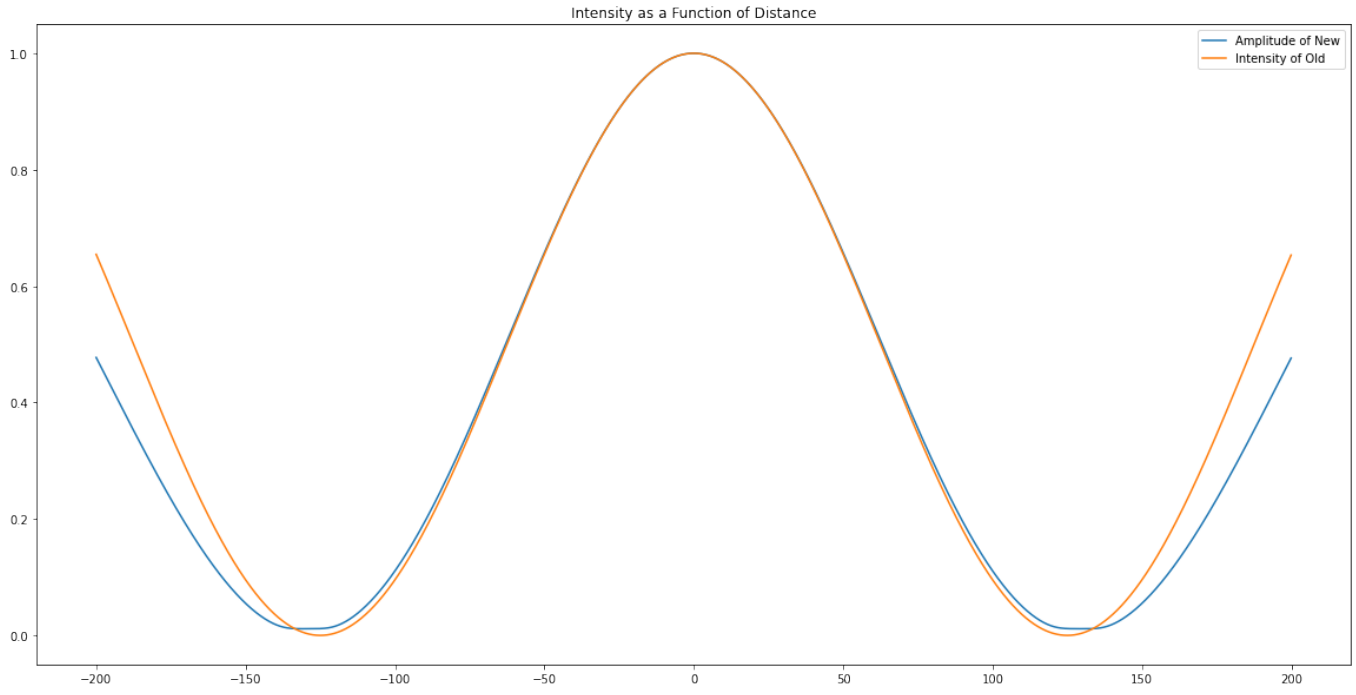
and to normalize it (since only the relative amplitudes matter), we're left with:

$$\delta(x_1)+\frac{1}{2}\delta(x_1-d)+\frac{1}{2}\delta(x_1+d). \tag{15}$$

We can verify this experimentally. Setting

```
slits = [
  [0*wavelength, 1*wavelength, wavelength, wavelength, 1],
  [-2*wavelength, 1*wavelength, wavelength, wavelength, 0.5],
  [2*wavelength, 1*wavelength, wavelength, wavelength, 0.5],
]
```

will give us the following plot, where we have plotted the intensity pattern of the previous aperture pattern for reference:



Notice that this is not perfect. This is because the Fourier Transform uses the far-field approximation, but the location of the first minimum isn't necessarily small enough.

Problem 2.2

We do a similar thing. Note that:

$$\cos^4(x)=\left(\frac{1}{2}(1+\cos(2x))\right)^2=\frac{1}{4}(1+2\cos(2x)+\cos^2(2x)) \tag{16}$$

Applying linearity of the fourier transform, we have

$$\begin{aligned}\hat{f}\left[\frac{1}{4}(1+2\cos(2ax)+\cos^2(2ax))\right] &= \frac{1}{4}\delta(x)+\frac{1}{4}\delta\left(x-\frac{2a}{2\pi}\right)+\frac{1}{4}\delta\left(x+\frac{2a}{2\pi}\right)+\frac{1}{4}\hat{f}\left[\cos^2(2ax)\right] \\ &= \frac{1}{4}\delta(x)+\frac{1}{4}\delta(x+d)+\frac{1}{4}\delta(x-d)+\frac{1}{4}\left(\frac{1}{2}\delta(x_1)+\frac{1}{4}\delta\left(x_1-2d\right)+\frac{1}{4}\delta\left(x_1+2d\right)\right) \\ &= \frac{3}{8}\delta(x)+\frac{1}{4}\delta(x\pm d)+\frac{1}{16}\delta(x\pm 2d)\end{aligned}$$

where we have let $a=\pi d$, and used the result from the previous problem. Normalizing this gives a slit in the middle that lets the full amplitude in, two slits located a distance d from the center that reduces the amplitude by $2/3$, and two slits located a distance $2d$ from the center that reduces the amplitude by $\frac{1}{6}$.

Problem 2.3

We want to find the Fourier transform:

$$\hat{f}[\cos^n(ax_2)] \tag{17}$$

However, multiplication in the x_2 domain is equivalent to convolution in the x_1 domain. We can apply the convolution theorem to say that:

$$\hat{f}[\cos^n(ax_2)] = \hat{f}[\cos(ax_2)]^n \tag{18}$$

where multiplication on the RHS is denoted by the convolution operator $\star$. We've seen that

$$\hat{f}[\cos(ax_2)] = \frac{1}{2}\left(\delta\left(x_1-\frac{a}{2\pi}\right)+\delta\left(x_1+\frac{a}{2\pi}\right)\right) \tag{19}$$

Note that convolution is associative and distributive. Using the fact that δ is the identity, we have the property:

$$\delta(x-a)\star\delta(x-b)=\delta(x-a-b) \tag{20}$$

Thus, we have:

$$\hat{f}[\cos(ax_2)]^n = \frac{1}{2^n}\left(\delta\left(x_1-\frac{a}{2\pi}\right)+\delta\left(x_1+\frac{a}{2\pi}\right)\right)^n$$

Note that $\frac{a}{2\pi}=\frac{d}{2}$. Expanding this using the distributive property, we see that each term contains k copies of $\delta(x-d/2)$ and $n-k$ copies of $\delta(x+d/2)$, which combines to give $\delta(x-kd/2+nd/2-kd/2)=\delta(x+(n/2-k)d)$ where $0\le k\le n$. If we label the $n+1$ slits as S_k, then S_k is located at a location of $(k-n/2)d$ and reduces the amplitude by $\frac{1}{2^n}\binom{n}{k}$. Here the $\binom{n}{k}$ comes in because there are $\binom{n}{k}$ terms in the convolution expansion that has k copies of the first term and $n-k$ copies of the second term (which corresponds to a unique slit).

Part Three

Problem 3.1

First, we make a few observations:

- The intensity pattern is symmetrical and centered at $x=0$: Therefore it probably consists of only cosines.
- The intensity never reaches 0, so the amplitude is always positive. We can then safely take the square root without worrying about reversibility issues.
- The number of peaks in between each period is 1, so it must mean that one frequency is exactly double the other. Looking at the period, we can conclude that two of the slits are located $\pm 5\lambda$ away from the center and $\pm 10\lambda$ frmo the center.

Using a data analysis tool such as Python or Excel, we find that the period is 102100.0λ.

Since the slits are at integer locations, we can write out a Fourier *series* by noting that each cosine would be in the form of:

$$\cos\left(d' \cdot \frac{2\pi}{\lambda D}x\right) \tag{21}$$

where d is an integer. Thus, let's clean up our data by:

- Taking the square root.
- Looking at only one period (to prevent edge effects from ruining the data)
- Plotting against $10\pi x/\lambda D$ instead of just x.

Taking a Fourier Series (i.e. by having $n = 2$), we get:

```
{y: a0 + a1*cos(w*x) + a2*cos(2*w*x) + b1*sin(w*x) + b2*sin(2*w*x)}

Parameter Value        Standard Deviation
a0        5.407222e-01 9.935952e-04
a1        1.419923e-01 1.439942e-03
a2        2.758724e-01 1.411563e-03
b1        -3.714775e-05 1.369781e-03
b2        -6.656304e-05 1.378570e-03
```

allowing us to reconstruct the sinusoidal wave. Taking a Fourier Transform, we get the aperture function. Here it is, for reference:

```
slits = [
      [0*wavelength, 1*wavelength, wavelength, wavelength, 3],
      [-5*wavelength, 1*wavelength, wavelength, wavelength, 0.45],
      [5*wavelength, 1*wavelength, wavelength, wavelength, 0.45],
      [-10*wavelength, 1*wavelength, wavelength, wavelength, 0.77],
      [10*wavelength, 1*wavelength, wavelength, wavelength, 0.77],
    ]
```

Problem 3.2

Similar to the previous problem, we "cut" off the interference pattern at a small angle (but big enough such that the distinctive behaviour of the curve is captured). Scaling x to be $\frac{2\pi}{\lambda D}x$, we can determine the Fourier coefficients to be:

```
Parameter Value        Standard Deviation
a0        2.045300e-01 1.285129e-04
a1        3.294887e-01 1.439479e-04
a2        2.371716e-01 8.086478e-05
a3        1.652102e-01 8.350614e-05
a4        6.081714e-02 2.142034e-04
```

Taking the Fourier Transform analytically, we get

$$0.0854\delta(x \pm 4) + 0.2070\delta(x \pm 3) + 0.2973\delta(x \pm 2) + 0.4130\delta(x \pm 1) + 0.5126\delta(x) \tag{22}$$

Plotting f_X against x forms a straight line ($r^2 = 0.998$).

Problem 3.3

Note that the aperture function here is actually a triangle:

$$\text{tri}(x_1/5) \tag{23}$$

The Fourier Transform from x_1 to x_2 gives:

$$\operatorname{sinc}^2\left(\frac{5x_2}{\lambda D}\right). \tag{24}$$