OPhO Invitational: Fluid Dynamics — Solutions

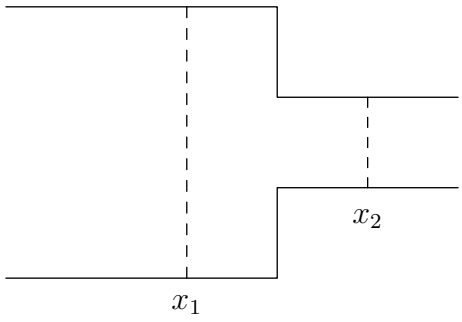
August 1, 2022

1 Introduction

No official solution was published. The solution below was worked on the HaslaLab port of D. Schroeder's fluid app, which the problem names, with the **default settings** kept as the problem asks (flow speed $u_0 = 0.100$, viscosity $\nu = 0.020$). The app is a D2Q9 lattice-Boltzmann (BGK) solver: every edge cell is reset each step to equilibrium at density 1 and velocity $(u_0, 0)$, and barriers use bounce-back. Everything is in lattice units (cell size 1, one step 1). All readings are the sensor's ρ, u_x and u_y.

2 Mass Flow Conservation (15 pts)

We will first investigate if this implementation of the Lattice-Boltzmann algorithm correctly simulates the conservation of mass.



1. Create a tube that either opens up or closes down very abruptly such as in the diagram above. Pick two locations x_1 and x_2 such that one is in the wider section and the other is in the more narrow section. Compute the flow $\frac{dm}{dt}$ past both these points. (10 pts)
2. Do the two mass flow rates agree with each other? If not, suggest a possible reason why. (5 pts)

Solution

Set-up. On the 200×80 grid two horizontal walls form a pipe: for $x < 100$ they are at $y = 20$ and $y = 60$ (39 fluid cells across), for $x \geq 100$ at $y = 30$ and $y = 50$ (19 cells), joined by vertical walls at $x = 100$. Fluid enters from the left edge and runs down the pipe. We chose $x_1 = 50$ in the wide part and $x_2 = 150$ in the narrow part.

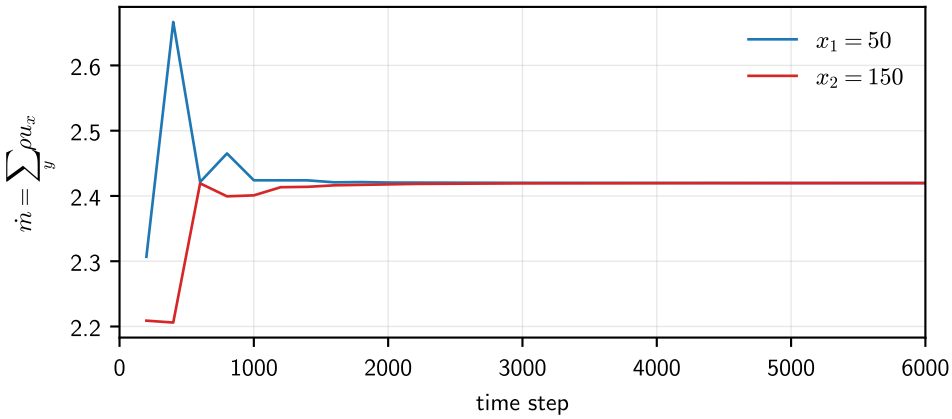
Measurement [10 pts]. The mass flow per unit depth is the integral of the mass flux across the section,

$$\frac{dm}{dt} = \int \rho\, u_x \, dy \approx \sum_y \rho(x,y)\, u_x(x,y)\, \Delta y. \tag{1}$$

We moved the sensor across each section one cell at a time, read ρ and u_x, set $u_x = 0$ on the walls (a bounce-back wall lies half a cell beyond the last fluid node) and integrated with the trapezoid rule. Once the flow is steady (after about 2000 steps, see the figure) the readings give

$$\left.\frac{dm}{dt}\right|_{x_1} = 2.417, \qquad \left.\frac{dm}{dt}\right|_{x_2} = 2.410 \qquad \text{(lattice units)}. \tag{2}$$

Reading every second cell gives 2.411 and 2.390, every fourth cell 2.389 and 2.319; the trapezoid error is larger in the narrow section, which has fewer cells. Taking these changes as the uncertainty, $dm/dt|_{x_1} = 2.417 \pm 0.006$ and $dm/dt|_{x_2} = 2.410 \pm 0.02$. Summing every cell exactly inside the simulation gives 2.4197 at both places, equal to ten decimal places.



Do they agree [5 pts]. In the steady state they agree within the uncertainties; the simulation conserves mass locally (collisions leave each cell's $\sum_i f_i$ unchanged and streaming only moves the distributions). A measurement can still disagree because of:

- **Reading before the steady state.** During the first ~ 2000 steps the two rates differ by up to 20% (figure). A lattice-Boltzmann fluid is slightly compressible, so mass piles up or drains from the pipe while the pressure gradient builds up.
- **Leaving out the density.** The densities in the two sections are $\rho \approx 1.127$ and 1.081, 4% apart (the pressure drops along the narrow pipe). Comparing volume flows $\int u_x \, dy$ instead of mass flows gives 2.147 and 2.238, 4% apart.
- **Coarse sampling.** The velocity profile is close to a parabola, so reading only a few points across the narrow section underestimates the integral (3% for every fourth cell).
- At higher speed or lower resolution the flow turns unsteady (vortex shedding) and a single reading fluctuates; then the Data record should be averaged in time.

3 Law of the Wall (25 pts)

The Law of the Wall provides an emperical relationship between the average velocity of a turbulent flow at some point and the distance to a wall (or fluid boundary). Namely,

$$u_x = \frac{u_T}{\kappa} \ln\left(\frac{\rho u_T}{\mu} y\right) + C, \tag{3}$$

where $u_T = \sqrt{\tau_W/\rho}$ is the shear velocity, $\tau_w = \mu \left.\dfrac{\partial u_x}{\partial y}\right|_{y=0}$ the wall shear stress and κ, C are constants. Studies have shown that κ ranges from 0.35 to 0.42. Using a straight line to model a wall, answer the following questions:

1. Is the relationship between u_x and y logarithmic? (10 pts)
2. What are the values of κ and C? (10 pts)
3. Does the value of κ agree with experiment? If it does not, suggest a possible reason why. (5 pts)

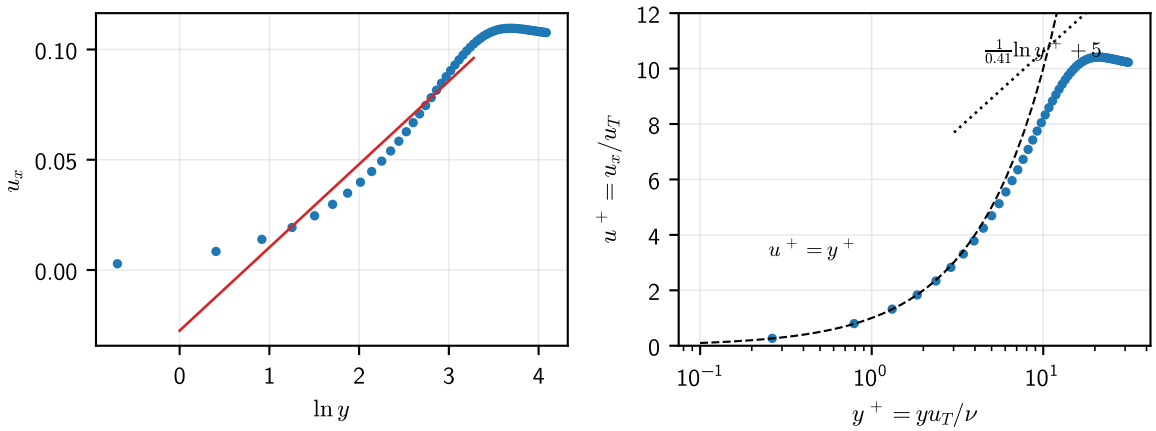
Solution

Set-up. On the 600×240 grid a horizontal wall was drawn across the whole tunnel at $y = 20$ and the flow was run for 30000 steps to steady state. A boundary layer grows along the wall; at $x = 200,\ 300,\ 400,\ 500$ the sensor was stepped up from the wall one cell at a time to read u_x. A bounce-back wall surface lies midway between the wall cell and the first fluid cell, so the distance from the wall is $y = n - \frac{1}{2}$ for the n-th cell. The boundary-layer thickness (where $u_x = 0.99U$) is $\delta_{99} = 18.5$–26.5 cells.

Shear velocity. A straight line through the three points next to the wall gives $\partial u_x/\partial y$; with $\mu = \rho\nu$,

$$u_T = \sqrt{\frac{\tau_w}{\rho}} = \sqrt{\nu \frac{\partial u_x}{\partial y}} = 0.0105\text{–}0.0120, \tag{4}$$

which defines wall units $y^+ = y u_T/\nu$ and $u^+ = u_x/u_T$.



1. Logarithmic? [10 pts] No. Plotting u_x against $\ln y$ (left, $x = 400$) shows no straight stretch. In wall units (right) the data follow the viscous-sublayer law $u^+ = y^+$ (dashed) over most of the layer ($y^+ \lesssim 5$), and even the edge of the layer only reaches $y^+ = 11$–15, short of the logarithmic region ($y^+ \gtrsim 30$, dotted). Over the first 1–7 cells a straight line fits with $R^2 = 0.9997$, a logarithm with $R^2 = 0.96$.

2. κ and C [10 pts]. Forcing (3) anyway, a fit $u_x = a \ln y + b$ over $1 \le y \le \delta_{99}$ gives $\kappa = u_T/a$ and $C = b - a\ln(u_T/\nu)$. The four stations give $\kappa = 0.301,\ 0.283,\ 0.279,\ 0.304$ and $C = -0.0008,\ -0.0025,\ -0.0033,\ -0.0024$ (lattice velocity units), so

$$\kappa = 0.29 \pm 0.01, \qquad C = -0.002 \pm 0.001 \ \ (C/u_T \approx -0.2), \tag{5}$$

the uncertainty being the scatter between stations. Restricting the fit to the inner 30% of the layer gives $\kappa \approx 0.55$ instead, itself a sign that the relationship is not logarithmic.

3. Agreement with experiment [5 pts]. No: 0.29 lies outside 0.35–0.42. The reasons:

- **The flow is not turbulent.** The law of the wall describes turbulent boundary layers. Here $Re_x = u_0 x/\nu = 1000$–2500, far below the $Re_x \sim 5 \times 10^5$ at which a flat-plate boundary layer turns turbulent. The layer is laminar (a Blasius profile) and has no logarithmic region at all.
- **The Reynolds number cannot be raised.** Lattice Boltzmann is only accurate for $u_0 \ll c_s = 1/\sqrt{3}$ (a Mach-number limit) and becomes unstable as ν is lowered, so this program cannot reach a large Re; and a two-dimensional flow at large Re is still not three-dimensional turbulence.
- **Resolution.** With only about 20 cells across the layer the slope at the wall, and hence u_T, is resolved coarsely.