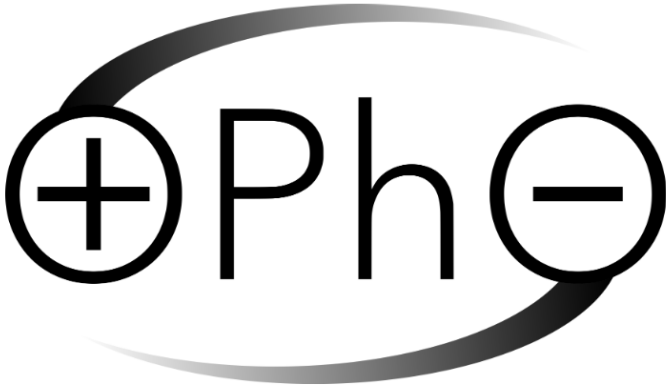
2023 Online Physics Olympiad: Invitational Contest



Experimental Exam Solutions

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Solution

No official solution was published. The solution below was worked out by HaslaLab with the organisers' own notebook functions (`system`, `rk4_step`, `rk2_step`, `get_distance`). Time is in the dimensionless units of the Lorenz equations.

Problem 1

What condition needs to be satisifed for σ, ρ, β in order for the ODE to be stable? (i.e. bounded orbits). Where are the attractors located?

Boundedness. Consider

$$V = x^2 + y^2 + (z - \sigma - \rho)^2. \tag{1}$$

Along the flow the xyz and xy terms cancel, leaving

$$\frac{dV}{dt} = -2\left[\sigma x^2 + y^2 + \beta\left(z - \frac{\sigma+\rho}{2}\right)^2\right] + \frac{\beta(\sigma+\rho)^2}{2}. \tag{2}$$

If $\sigma > 0$ and $\beta > 0$ the bracket is positive definite, so $dV/dt < 0$ outside an ellipsoid. Every orbit therefore enters a bounded level set of V in finite time and never leaves: **for $\sigma > 0$, $\beta > 0$ the orbits are bounded for any ρ.** Phase-space volume also shrinks at the rate $\nabla \cdot \mathbf{f} = -(\sigma + 1 + \beta) < 0$, so orbits collapse onto an attractor of zero volume.

Fixed points and attractors. Setting $\dot{x} = \dot{y} = \dot{z} = 0$ gives the origin O and, for $\rho > 1$,

$$C_\pm = \left(\pm\sqrt{\beta(\rho-1)},\ \pm\sqrt{\beta(\rho-1)},\ \rho - 1\right). \tag{3}$$

For $\rho < 1$ the origin is the only, and stable, attractor. The linearisation about $C_\pm$ has the characteristic equation $\lambda^3 + (\sigma+\beta+1)\lambda^2 + \beta(\sigma+\rho)\lambda + 2\sigma\beta(\rho-1) = 0$, and the Routh–Hurwitz condition shows that, for $\sigma > \beta + 1$, $C_\pm$ are stable only for

$$1 < \rho < \rho_H = \frac{\sigma(\sigma+\beta+3)}{\sigma-\beta-1}. \tag{4}$$

For $\sigma = 10$, $\beta = 8/3$, $\rho_H = 470/19 \approx 24.74$. At $\rho = 28 > \rho_H$ all three fixed points are unstable (the origin is a saddle; at $C_\pm$ the eigenvalues are -13.85 and $0.094 \pm 10.19i$) and orbits settle on the strange (Lorenz) attractor, whose two wings wind around

$$C_\pm = (\pm 6\sqrt{2},\ \pm 6\sqrt{2},\ 27) \approx (\pm 8.49,\ \pm 8.49,\ 27). \tag{5}$$

Problem 2

Suppose you change the parameter ρ slightly above its original value, and keep σ and β at their original values. How does the system behavior change? In particular, how does the position of the attractors change, and what happens to the system's stability?

Now suppose you keep ρ at its original value, and change σ and β to values slightly different from their original ones. Again, describe the changes in the system behavior, the position of the attractors, and the system's stability.

Raising ρ. By (3) the centres of the wings move up ($z = \rho - 1$, $dz/d\rho = 1$) and outwards ($|x| = |y| = \sqrt{\beta(\rho-1)}$), and the whole attractor grows. Since ρ_H does not depend on ρ, $C_\pm$ become more unstable: the real part of their complex eigenvalues is 0.094, 0.147, 0.198 at $\rho = 28$, 30, 32, and the maximal Lyapunov exponent measured as in Problem 3 rises from 0.909 ± 0.012 to 0.954 ± 0.004 and 0.996 ± 0.008. The system becomes more chaotic.

Changing σ and β. The height $\rho - 1$ of the centres does not change, and their x, y coordinates $\pm\sqrt{\beta(\rho-1)}$ depend on β only (± 9 for $\beta = 3$, ± 8.22 for $\beta = 2.5$); σ does not move them at all. What changes is ρ_H: 24.75 and 25.0 for $\sigma = 9$ and 11, 23.85 and 26.67 for $\beta = 2.5$ and 3. For small changes $\rho = 28$ stays above ρ_H, so the motion stays chaotic and only the maximal Lyapunov exponent shifts ($\sigma = 9$:

0.890, $\sigma = 11$: 0.912, $\beta = 2.5$: 0.872, $\beta = 3$: 0.939). Changes that push ρ_H towards 28 (such as larger β) weaken the chaos until, once $\rho_H > 28$, orbits converge to C_+ or C_-.

Problem 3

Compute the Maximal Lyapunov Exponent (MLE) for a Lorenz system of these parameters:

- $\sigma = 15.6$
- $\rho = 35.4$
- $\beta = 3.13$

What does this imply about the system? See if you can estimate uncertainties!

Following two orbits for a long time does not work, because $d(t)$ saturates at the size of the attractor; we use Benettin's renormalisation instead. A reference orbit is run to $t = 50$ so that it lies on the attractor, and a second orbit is started a distance $d_0 = 10^{-8}$ away. Both are advanced with `rk4_step` ($dt = 0.01$) for $\tau = 1$, the separation d_k is measured with `get_distance`, and the second orbit is pulled back to distance d_0 along the same direction. After $n = 200$ such steps

$$\lambda \approx \frac{1}{n\tau} \sum_{k=1}^{n} \ln \frac{d_k}{d_0}. \tag{6}$$

Ten runs from different random starting points gave

$$1.186,\ 1.179,\ 1.150,\ 1.170,\ 1.165,\ 1.174,\ 1.163,\ 1.143,\ 1.170,\ 1.159$$

so

$$\lambda_{\max} = 1.17 \pm 0.01 \tag{7}$$

(standard deviation 0.013, standard error of the mean 0.004). As a check, the same code gives 0.909 ± 0.012 for $\sigma = 10$, $\rho = 28$, $\beta = 8/3$, close to the accepted 0.906.

Since $\lambda_{\max} > 0$ the system is chaotic: nearby orbits separate on average like $e^{1.17t}$, so an initial error grows by a factor e every $1/\lambda \approx 0.86$ time units. Consistently, $\rho_H = 29.55 < 35.4$ here, so all the fixed points are unstable.

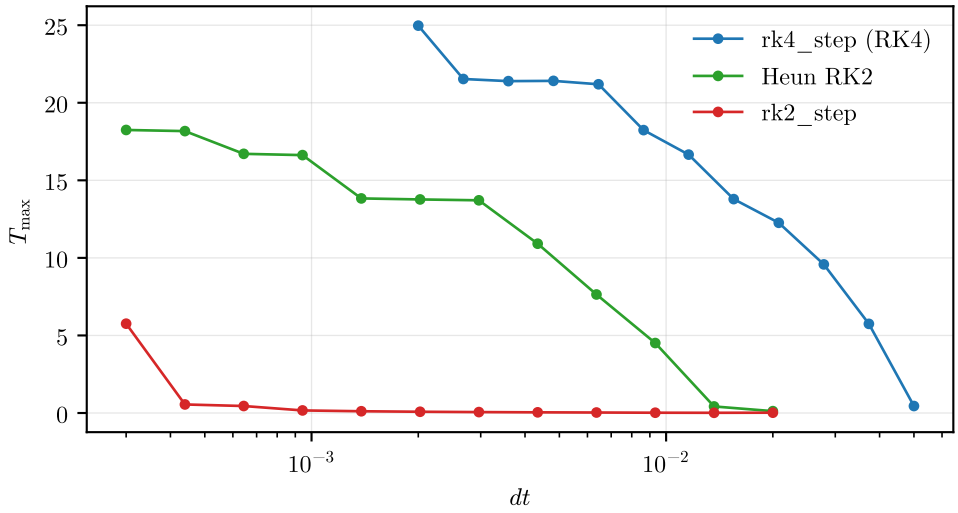
Problem 4

Call $T_{\max}$ the maximal time for when a simulation is accurate to 99% of reality. A simulation is characterized by the specific numerical solver and the time step dt.

Using `rk4_step`, estimate $T_{\max}$ at various values of dt, given the initial point of $(1, 1, 1)$. Make a plot.

Then do the same thing with `rk2_step`. What differences do you notice?

For "reality" we take an eighth-order Dormand–Prince solution (scipy DOP853) with tolerance 2×10^{-14}. It agrees with a run at tolerance 3×10^{-14} to 10^{-3} up to $t = 38.6$, so it can be trusted that far. $T_{\max}$ is the first time at which the relative error $|\mathbf{r}_{dt} - \mathbf{r}|/|\mathbf{r}|$ exceeds 1%.



- **`rk4_step`:** $T_{\max} = 0.45$ at $dt = 0.05$, 9.6 at 0.028, 16.7 at 0.0116 and 25.0 at 0.002. The global error of RK4 grows like $dt^4 e^{\lambda t}$, so one expects $T_{\max} \approx \frac{1}{\lambda} \ln \frac{0.01}{C\, dt^4}$: a straight line against $\ln dt$ with slope $-4/\lambda \approx -4.4$. The measured slope is about -5. The steps appear because the error does not grow evenly; it jumps when the orbit passes near the saddle at the origin and switches wings.
- **`rk2_step`:** $T_{\max}$ is essentially zero (5.8 at $dt = 3 \times 10^{-4}$, below 1 for every larger dt). The notebook's `rk2_step` is $\mathbf{s} + dt\,\mathbf{f}(\mathbf{s} + dt\,\mathbf{f}(\mathbf{s}))$, whose Taylor expansion has 1 instead of $\frac{1}{2}$ in front of the dt^2 term: it is really only **first-order** accurate. A genuine second-order method, Heun's $\mathbf{s} + \frac{dt}{2}[\mathbf{f}(\mathbf{s}) + \mathbf{f}(\mathbf{s} + dt\,\mathbf{f}(\mathbf{s}))]$, gives $T_{\max} = 4.5$ at $dt = 0.0093$, 13.7 at 0.003 and 18.2 at 3×10^{-4}, growing with slope $\approx -2/\lambda$, more slowly than RK4.

So RK4 reaches a given $T_{\max}$ with a much larger dt than a second-order method, and halving dt buys about $4 \ln 2/\lambda \approx 3.1$ time units with RK4 against about 1.5 with a second-order scheme. With any method $T_{\max}$ grows only logarithmically as dt shrinks, because in a chaotic system every error is amplified like $e^{\lambda t}$.

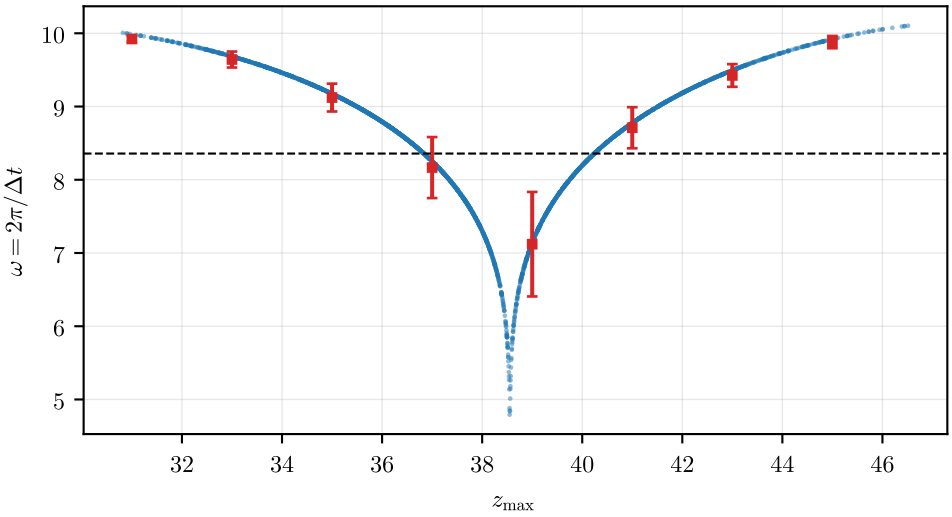
Problem 5

Let $\sigma = 10$, $\rho = 28$, $\beta = 8/3$. What is the average angular frequency ω_0? Report with uncertainty. How does this change with position?

Each loop around C_+ or C_- passes exactly one maximum of z. We integrated with `rk4_step` ($dt = 0.002$) from $t = 20$ to 2020, located the maxima of z (refining their times by parabolic interpolation), and took $2\pi/\Delta t$ between neighbouring maxima as the angular frequency of each loop. Over 2660 loops

$$\omega_0 = \frac{2\pi N}{t_N - t_0} = 8.36 \pm 0.01 \tag{8}$$

(mean period 0.752). The uncertainty is the standard error of ω_0 computed separately on 20 blocks of the data.



The plot shows how it changes with position; the horizontal axis is the maximum of z on that loop, a measure of its size. Small loops close to $C_\pm$ ($z_{\max} \approx 31$) turn at $\omega \approx 10$, close to the imaginary part 10.19 of the linearised eigenvalues. As the orbit spirals outwards the loops slow down, reaching $\omega \approx 5$ for loops that graze the saddle at the origin ($z_{\max} \approx 38.5$), where the orbit lingers. Still larger loops, which carry the orbit to the other wing, speed up again to $\omega \approx 10$ at $z_{\max} \approx 46$. The angular frequency is therefore not constant: it falls as the orbit moves away from $C_\pm$ towards the origin.